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Moderate Deviations Principle for stochastic system with fixed delay in strong topology

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Résumé : Dans cet article, nous étudions un principe de déviations modérées vérifié par les systèmes stochastiques avec mémoire. Il s'agit d'un système décrit par l'équation différentielle stochastique avec mémoire suivante :

$$\begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dt + \sqrt{\varepsilon}\sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dW_t & t \in (0, \infty) \\ X_t^\varepsilon = \psi(t) & \text{t} \in [-\tau, 0] \end{cases}$$

où W est un mouvement brownien standard $d-$dimensionnel, b et σ vérifient les conditions lipschitziennes et sont à croissance linéaire ; ψ est une fonction continue donnée définie sur $[-\tau, 0]$.L'étude se fait avec la topologie hölderienne.
On présente ici l'étude des processus $Y^\varepsilon := (Y_t^\varepsilon = \dfrac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}})_{t\in[-\tau,m]}$ et $Z^\varepsilon := (Z_t^\varepsilon = \dfrac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}h(\varepsilon)})_{t\in[-\tau,m]}$ dans le cadre du Théoreme de la limite centrale et du Principe de déviation modérée respectivement. On utilisera la propriété de l'équivalence exponentielle et l'inégalité de transportation de Talagrand à cette fin.

Mots-clés : principe de déviations modérées , systèmes stochastiques, retard, inégalité de transportation de Talagrand, équivalence exponentielle, norme hölderienne.

Abstract : In this paper, we develop a moderate deviations principle for stochastic system with memory driven by small multiplicative white noise. The system is described by :

$$\begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dt + \sqrt{\varepsilon}\sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dW_t & t \in (0, \infty) \\ X_t^\varepsilon = \psi(t) & \text{t} \in [-\tau, 0] \end{cases}$$

where W is a $d-$dimensional brownian motion, b and σ are Lipschitzian and have a linear growth ;ψ is a given continuous function on $[-\tau, 0]$. One work in Hölder topology.
One focus on the study of $Y^\varepsilon := (Y_t^\varepsilon = \dfrac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}})_{t\in[-\tau,m]}$ and $Z^\varepsilon := (Z_t^\varepsilon = \dfrac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}h(\varepsilon)})_{t\in[-\tau,m]}$ in order to get a central limit theorem and a moderate deviation principle respectively. For the proof, we will use a theorem of exponential equivalence and Talagrand's transportation inequality.

Keywords : moderate deviations principle, stochastic system, delay, Talagrand's transportation inequality, exponential equivalence, hölderian norm.

Introduction

Let $W_t := (W_t^1, W_t^2, ..., W_t^l)$ denote a standard l-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with $W_0 = 0$ and W_t takes values in $\mathbb{R}^d$, $\Omega = \mathbb{C}_0([0, m], \mathbb{R}^l)$ equipped with the usual topology of uniform convergence defined by the norm

$$\|f\|_\infty = \sup_{0\leq t\leq m} |f(t)|.$$

Let $b : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^l$ be Borel measurable functions with linear growth.

Let $\tau > 0$ be a fixed delay, and ψ be a given continuous function on $[-\tau, 0]$.

Consider the stochastic delay differential equation (sdde)

$$\begin{cases} dX_t &= b(t, X_t, X_{t-\tau})dt \quad t \in (0, \infty) \\ X_t &= \psi(t) \quad\quad\quad\quad\quad t \in [-\tau, 0] \end{cases} \tag{1}$$

and the perturbed stochastic delay differential equation associated

$$\begin{cases} dX_t^\varepsilon &= b(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dt + \sqrt{\varepsilon}\sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dW_t \quad t \in (0, \infty) \\ X_t^\varepsilon &= \psi(t) \quad\quad\quad\quad\quad t \in [-\tau, 0] \end{cases} \tag{2}$$

We study deviation of X^ε from the deterministic solution X, that is the asymptotic behavior of the trajectory of

$$Z^\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}h(\varepsilon)}(X_t^\varepsilon - X_t)$$

where $h(\varepsilon)$ is some deviation scale which strongly influences the asymptotic behavior of Z^ε. If one put $h(\varepsilon) = \dfrac{1}{\sqrt{\varepsilon}}$, one get large deviations (LD) estimates. T. Zhang and S. E.A Mohammed in [12] give the Large Deviations principle (LDP) for the family of $\{\mu_\varepsilon; \varepsilon > 0\}$ law of $\{X^\varepsilon; \varepsilon > 0\}$

in uniform topology. In our recent works in S. H. Randriamanirisoa and T.J. Rabeherimanana [8], we provide an extension of the LDP to Hölder space if we consider the case $Z \equiv 0$ and $Y \equiv 0$.

In the classical work, Friedlin-Wentzell [5] studied the Large deviations for small noise, limit of stochastic reaction-diffusion equations.
If one consider $h(\varepsilon) \equiv 1$, we are in the case of the Central Limit Theorem(CLT).
Set $Y^{\varepsilon}(t) := \frac{1}{\sqrt{\varepsilon}}(X^{\varepsilon} - X)$. One shall show that Y^{ε} converge as $\varepsilon \downarrow 0$ to a random field. Then, we will study Moderate Deviations (MD), that is when deviation scale satisfies $h(\varepsilon) \longrightarrow +\infty$, and $\sqrt{\varepsilon}h(\varepsilon) \longrightarrow 0$ as $\varepsilon \downarrow 0$.

In this paper, we establish its moderate deviation principle (MDP) in the context of Höoder norm. For this end, we will use Talagrand's transportation inequality on path space established in Djellout and al. [4] and measure concentration see in Bobkov and Götze [1], Djellout and al. [4], and Villani [9].
The paper is organized as follow : in the first section, we give basic settings and some useful general results. We find the main results in section 2. In section 3 we will provide some useful results. The last section is devoted to prove the results.

1 Basic Settings

Let us consider (2). W_t is defined on some well filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We always assume :

(H_1) : σ, b are locally Lipschitzian.
That means, there exists some constants L_b and L_σ such that for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ and $t \in [0, \infty)$

$$\|\sigma(., x_1, y_1) - \sigma(., x_2, y_2)\|_{\mathbb{R}^d \otimes \mathbb{R}^l} \leq L_\sigma(|x_1 - x_2| + |y_1 - y_2|) \tag{3}$$

$$\|b(., x_1, y_1) - b(., x_2, y_2)\|_{\mathbb{R}^d} \leq L_b(|x_1 - x_2| + |y_1 - y_2|) \tag{4}$$

(H_2) : $\sigma(., x, y), b(., x, y)$ are uniformly continuous in x, y $\in \mathbb{R}^d$ *on* $[0, \infty)$, that means :

$$\lim_{s\to t} \sup_{x,y\in\mathbb{R}^d} |\sigma(s, x, y) - \sigma(t, x, y)| = 0.$$

$$\lim_{s\to t} \sup_{x,y\in\mathbb{R}^d} |b(s, x, y) - b(t, x, y)| = 0.$$

(H_3) : b is C^1 .

(H_4) : There exists $C > 0$ such that

$$\max\{tr(\sigma\sigma * (t, x, y)), \langle x, b(t, x, y)\rangle, \langle y, b(t, x, y)\rangle\} \leq C(1 + |x|^2 + |y|^2)$$

where $\langle x, y\rangle = x.y$ is the inner product with $\langle x, x\rangle = |x|^2$ and

(H_5) : There exist a constant K such that

$$\sup\{|b(t,x,y)+\sigma(t,x,y)|\} \leq K(1+|x|+|y|) \tag{5}$$

(2) has a unique non-explosive solution denoted by $X^\varepsilon = (X^\varepsilon_t)_{t\in[-\tau,m]}$ under the conditions (H_1), (H_2). Let $\mathcal{H}$ denote the Cameron-Martin space associated with the Brownian motion W.

$$\mathcal{H} = \{h(t) = \int_0^t \dot{h}(s)ds : \dot{h} \in L^2([0,T]), h \quad \text{is absolutely continuous}\}$$

When ε goes to 0, we get :

$$\lim_{\varepsilon\to 0} P(\sup_{t\in[-\tau,\infty)} |X^\varepsilon_t - X^0_t| > R) = 0, \qquad \forall \;\; R > 0$$

where X^0 is the solution of the non-perturbed dynamical system (1). Let μ^ε the law of X^ε, μ^ε converges tightly to the degenerated measure δ_{X^0}.
Let $C_\psi([-\tau,m],\mathbb{R}^d)$ be the Banach space of continuous functions $f : [-\tau,m] \to \mathbb{R}^d$, with $f(t) = \psi(t)$ for $t \in [-\tau, 0]$; equipped with the sup-norm

$$\|f\| := \sup_{t\in[0,m]} |f(t)|.$$

Let $C_\psi([0,m],\mathbb{R}^d)$ be the Banach space of continuous functions $g : [0,m] \to \mathbb{R}^d$, with $g(0) = 0$. For a precise asymptotic estimate of deviations of X^ε from the dynamical system X^0, T. Zhang and S.E.A Mohammed in [12] claim that the law of $\{X^\varepsilon_t, \varepsilon > 0\}$ satisfies a LDP in uniform topology on $C_\psi([-\tau,m],\mathbb{R}^d)$ with the good rate function given by

$$I_{LDP}(f) = \begin{cases} \inf\{\frac{1}{2}\int_0^m |\dot{g}(s)|^2 ds; & g \in C_\psi([0,m],\mathrm{R}^l), \\ & \text{g absolutely continuous}, F(g) = f\} \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

with F is solution to

$$\begin{cases} F(h)(t) &= \int_0^t b(s,F(h)(s),F(h)(s-\tau))ds + \int_0^t \sigma(s,F(h)(s),F(h)(s-\tau))dh(s) \quad t \in (0,+\infty) \\ F(h)(t) &= \psi(t) \qquad t \in [-\tau, 0] \end{cases} \tag{7}$$

Otherwise, $I_{LDP} = +\infty$.

The subject of this paper is the problem of MDP for $\{X^\varepsilon_t\}_{t\in[-\tau,m]}$ in Hölder norm.

Let $\alpha > 0$ and denote by $C^\alpha_0([-\tau,m],\mathbb{R}^d)$ the set of $\alpha-$ Hölder norm continuous functions i.e of continuous functions $f : [-\tau,m] \longrightarrow \mathbb{R}^d$ such that $f(t) = 0$, for $t \in [-\tau, 0]$, and

$$\|f\|_\alpha = \sup_t |f(t)| + \sup_{|s-t|\leq 1, s\neq t} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty$$

The norm $\|.\|_\alpha$ is called the $\alpha-$Hölder norm. It is well known that the trajectories of X are $\alpha-$ Hölder continuous for $\alpha \in [0, \frac{1}{2}[$.

Let $C_\psi^\alpha([-\tau, m], \mathbb{R}^d)$ denote the set of $\alpha-$ Hölder norm continuous functions i.e of continuous functions $f : [-\tau, m] \longrightarrow \mathbb{R}^d$ such that $f(t) = \psi(t)$, if $t \in [-\tau, 0]$ and

$$\|f\|_\alpha = \sup_t |f(t)| + \sup_{|s-t|\leq 1, s\neq t} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty.$$

Define the hölderian modulus of continuity of f by

$$\omega_\alpha(f, \delta) = \sup_{0<|t-s|\leq\delta; s,t,\in[0,m]} \frac{|f(t) - f(s)|}{|t - s|^\alpha}$$

$C_\psi^{\alpha,0}([-\tau, m], \mathbb{R}^d)$ denote a closed subspace of $C_\psi^\alpha([-\tau, m], \mathbb{R}^d)$ defined by $C_\psi^{\alpha,0}([-\tau, m], \mathbb{R}^d) = \{f \; in C_\psi^\alpha([-\tau, m], \mathbb{R}^d); \lim_{\delta\to 0} \omega_\alpha(f, \delta) = 0\}$ is separable.
Set

$$Y^\varepsilon := (Y_t^\varepsilon)_{t\in[-\tau,\infty)} := (\frac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}}) \tag{8}$$

and

$$Z^\varepsilon := (Z_t^\varepsilon)_{t\in[-\tau,\infty)} \text{ with } Z_t^\varepsilon := \frac{Y_t^\varepsilon}{h(\varepsilon)} \tag{9}$$

Here, we shall study the asymptotic behaviour of : Z^ε where

$$h(\varepsilon) \longrightarrow +\infty \quad \text{and} \quad \sqrt{\varepsilon} h(\varepsilon) \longrightarrow 0, \quad \text{as} \quad \varepsilon \longrightarrow 0 \tag{10}$$

Through this paper, we always assume that $h(\varepsilon)$ satisfies (10).

2 Main result

In addition to $(H_1) - (H_5)$, one assume that b is differentiable with respect to the second and last variable. And $\frac{\partial b}{\partial x}, \frac{\partial b}{\partial y}$ are also uniformly Lipschitz. More precisely, There exists some constants $L_{\partial_x b}$ and $L_{\partial_y b}$ such that :

$$|\frac{\partial b}{\partial x}(t, u, y) - \frac{\partial b}{\partial x}(t, v, y)| \leq L_{\partial_x b}|u - v| \tag{11}$$

$$|\frac{\partial b}{\partial y}(t, x, r) - \frac{\partial b}{\partial y}(t, x, s)| \leq L_{\partial_y b}|r - s| \tag{12}$$

$\forall t \in \mathbb{R}^+, u, v, x, y, r, s \in \mathbb{R}^d$.
Combined with the uniform Lipschitz continuity of b, one obtain that :

$$|\frac{\partial b}{\partial x}(t, u, y)| \leq L_b$$

$$|\frac{\partial b}{\partial y}(t,x,r)| \leq L_b$$

$\forall t \in \mathbb{R}^+, u, x, y, r \in \mathbb{R}^d$.

The purpose of this paper is to prove results concerning CLT and MDP which are satisfied by Y_t^ε and Z_t^ε respectively in Hölder space.

First, let us recall a LDP for X^ε in Hölder norm. This result is an adapted version of the result found in our previous work [8] .

Theorem 1 (LDP) : *Let μ_ε be the law of X^ε solution of (1) on $C_\psi([-\tau, m]; \mathbb{R}^d)$, equipped with the norm $\|.\|_\alpha$. The family $\{\mu_\varepsilon, \varepsilon > 0\}$ satisfies a LDP with the following rate function*

$$\tilde{I}^*(f) = \lim_{a\to 0} \inf_{\rho\in B_\alpha(f,a)} \tilde{I}(\rho)$$

where $\tilde{I}(\rho) = \inf\{\frac{1}{2}\int_0^m |\dot{h}(s)|^2 ds; h \in \mathcal{H} : F(h, r, u) \equiv \rho\}$
with F is solution to the differential delayed equation :

$$\begin{cases} F(h)(t) = & F(h)(0) + \int_0^t b(s, F(h)(s)), F(h)(s-\tau))ds \\ & + \int_0^t \sigma(s, F(h)(s), F(h)(s-\tau))\dot{h}(s)ds \qquad t \in [0,\infty) \\ F(h)(t) = & \psi(t) \qquad t \in [-\tau, 0] \end{cases}$$

That is ,

i) *For any closed subset $C \subset C_\psi^{\alpha,0}([-\tau, m]; \mathbb{R}^d)$*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \leq -\inf_{f\in C} \tilde{I}^*(f).$$

ii) *For any open subset $G \subset C_\psi^{\alpha,0}([-\tau, m]; \mathbb{R}^d)$*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \leq -\inf_{f\in G} \tilde{I}^*(f).$$

where $\tilde{I}^*$ is a good rate function with respect to the topology of $C_\psi^{\alpha,0}([-\tau, m]; \mathbb{R}^d), 0 \leq \alpha < \frac{1}{2}$.

Proof : The proof is a particular case of the main result in our paper [8] with $Y \equiv 0$ and $Z \equiv 0$.

Next result is about the CLT.

Theorem 2(Central Limit Theorem) : *Assume $(H_1) - (H_5)$. Let suppose $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial y}$ and σ verify all assumptions before. Then, $\forall \alpha \in [0, \frac{1}{4})$, $r \geq 1$, the random field Y^ε converge in L^r to a random field $Y \equiv Y^0$ as $\varepsilon \to 0$ on $C_\psi^\alpha([0, m], \mathbb{R}^d)$ determined by :*

$$\begin{cases} dY_t &= [\frac{\partial b}{\partial x}(t, X_t, X_{t-\tau})Y_t + \frac{\partial b}{\partial y}(t, X_t, X_{t-\tau})Y_{t-\tau}]dt + \sigma(t, X_t, X_{t-\tau})dW_t \quad t \in (0,\infty) \\ Y_t &= 0 \qquad t \in [-\tau, 0] \end{cases} \tag{13}$$

where $\frac{\partial b}{\partial x} = (\frac{\partial b}{\partial x_i})_{1\leq i\leq d}$ *and* $\frac{\partial b}{\partial y} = (\frac{\partial b}{\partial y_j})_{1\leq j\leq d}$.

By theorem 4 in the next section and in view of $\sqrt{\varepsilon}h(\varepsilon) \to 0, h(\varepsilon) \to \infty$, as ε tends to 0. One see that Z^ε obeys LDP on $C^\alpha_\psi([0,m],\mathbb{R}^d)$ with speed $h^2(\varepsilon)$, and the rate function

$$I_{MDP}(f) = \inf\{\frac{1}{2}\int_0^m |\dot{h}(t)|^2 dt, h \in \mathcal{H}, G(h) = f\} \tag{14}$$

where G is solution to the following deterministic equation :

$$\begin{cases} dG(h)(t) = & \frac{\partial b}{\partial x}(t, F(h)(t), F(h)(t-\tau))G(h)(t)dt + \frac{\partial b}{\partial y}(t, \quad F(h)(t), F(h)(t-\tau))G(h)(t-\tau)dt \\ & +\sigma(t, F(h)(t), F(h)(t-\tau))\dot{h}(t)dt \qquad t \in (0,\infty) \\ G(h)(t) = & 0 \qquad t \in [-\tau, 0] \end{cases}$$

where F is solution to (7). Hence, formally one get the following result.
Theorem 3(MDP) :*Assume* $(H_1) - (H_5)$ *. Let suppose* σ *is continuous,* $b, \partial_x b, \partial_y b$ *and* σ *satisfy assumptions above.*
Then, for all $\alpha \in [0, \frac{1}{4})$ *,* Z^ε *obeys a LDP on* $C^\alpha_\psi([0,m],\mathbb{R}^d)$ *with speed* $h^2(\varepsilon)$ *with rate function* I_{MDP} *defined by* (14).

3 Useful results

Let us recall some useful results.
Theorem 4(found in [3]) : *If an LDP with a good rate function I(.) holds for the probability measure* $\{\mu_\varepsilon, \varepsilon > 0\}$ *which are exponentially equivalent to* $\{\tilde{\mu}_\varepsilon, \varepsilon > 0\}$*, then the same LDP holds for* $\tilde{\mu}_\varepsilon$.
Lemma 5 : *Assume* (5).
Then, for any $p \geq 2$*, there exist* C *such that :*

$$\mathbb{E}|X^\varepsilon(t)|^p \leq C(1 + \sup|\psi(t)|^p)$$

Proof : One use a similar argument adapted as in Walsh [10], Theorem 3.2.
Proposition 6 : *Assume* (5) *and* ψ *bounded.*
Then, for all $p \geq 2$*, there exist* $C(p, K, \partial_x b, \partial_y b, T, \psi)$ *such that :*

$$\mathbb{E}(|X^\varepsilon - X|^T_\infty)^p \leq \varepsilon^{\frac{p}{2}} C(p, K, \partial_x b, \partial_y b, T, \psi) \longrightarrow 0 \ \ as \ \ \varepsilon \to 0$$

Proof : One use a similar argument adapted as in Wang and Zhang [11], Proposition 3.2.
We will use Talagrand's transportation inequality.
Let

$$W_p(\mu,\nu) := \inf(\int\int d(x,y)^p d\pi(x,y))^{\frac{1}{p}} \quad 1 \leq p \leq +\infty$$

L^p-Wasserstein distance between μ, ν two probability measures on E.

μ satisfies Talagrand's T_2-transportation inequality $T_2(C_T)$ on (E,d) if

$$\forall \nu : W_2(\nu,\mu)^2 \leq 2C_T H(\nu|\mu)$$

where $H(\nu|\mu)$ is Kullback-Leibler information or relative entropy of ν with respect to μ.

$$H(\nu|\mu) = \begin{cases} \int \log \frac{d\nu}{d\mu} d\nu & \nu << \mu \\ +\infty & \text{otherwise} \end{cases}$$

Consider $E = L^2([0,T];\mathrm{R}^d) = \{\phi : [0,T] \longrightarrow \mathrm{R}^d \text{ measurable};\ \|\phi\|_2^2 = \int_0^T |\phi(t)|^2 dt < +\infty\}$ equipped with the distance $d_2(\phi_1,\phi_2) := \|\phi_1 - \phi_2\|_2$. Let μ^ε the law of $X^\varepsilon = (X_t^\varepsilon)_{t\in[0,T]}$ a probability measure on E .

Lemma 7 : *Assume that σ, b are locally lipschitz functions, σ bounded such that :*

$$\frac{1}{2}(|(\sigma(s,x,y)-\sigma(s,u,v))(\sigma(s,x,y)-\sigma(s,u,v))^*| + \langle x-u, b(s,x,y)-b(s,u,v)\rangle + \langle y-v, b(s,x,y)-b(s,u,v)\rangle) \leq L(|x-u|^2+|y-v|^2)$$

Then, for all $\varepsilon \in [0,1]$, μ^ε the law of $X^\varepsilon := (X_t^\varepsilon)_{t\in[0,T]}$ satisfies on $(L^2([0,T];\mathbb{R}^d); d_2)$ T_2-Talagrand's inequality $T_2(\varepsilon C_T)$, where

$$C_T := \frac{\|\sigma\|_\infty^2(\exp((\delta+2L)T)-1)}{\delta(\delta+2L)}$$

$\delta > 0$ is taken arbitrary, and

$$\begin{aligned} \|\sigma\|_\infty &= \sup_{(t,x,y)\in\mathrm{R}^+\times\mathrm{R}^d\times\mathrm{R}^d, v\in\mathrm{R}^d, \|v\|=1} \|\sigma(t,x,y)v\| \\ &\leq \sqrt{\sup_{u\in\mathrm{R}^+\times\mathrm{R}^d\times\mathrm{R}^d} tr(\sigma\sigma^*(u))} \\ &\leq \sqrt{M} \end{aligned}$$

Proof : One use a similar argument as in Ma, Wang and Wu [7] lemma 4.2.

Lemma 8 : *Assume that σ, b are locally Lipschitzian, and σ bounded such that :*

$$\frac{1}{2}(|(\sigma(s,x,y)-\sigma(s,u,v))(\sigma(s,x,y)-\sigma(s,u,v))^*| + \langle x-u, b(s,x,y)-b(s,u,v)\rangle + \langle y-v, b(s,x,y)-b(s,u,v)\rangle) \leq L(|x-u|^2+|y-v|^2)$$

Then $\forall \varepsilon \in (0,1], r > 0$

$$P((\int_0^T |Y_t^\varepsilon|^2 dt)^{\frac{1}{2}} - \mathbb{E}(\int_0^T |Y_t^\varepsilon|^2 dt)^{\frac{1}{2}} \geq r) \leq \exp\{-\frac{r^2}{2C_T}\}$$

where $C_T = \dfrac{\|\sigma\|_\infty^2 (e^{(\delta+2L)T} - 1)}{\delta(\delta + 2L)}$

Proof : One use a similar adapted argument as in Ma, Wang and Wu [7] Lemma 4.3. Set $\Phi(\varphi) = \|\dfrac{\varphi - X^0}{\sqrt{\varepsilon}}\|_\infty$. Φ is a lipschitz function on $(L^p([0,m];\mathbb{R}^d); d_2)$ with the lipschitz constant $\dfrac{1}{\sqrt{\varepsilon}}$. Since $\|Y^\varepsilon\|_\infty = \Phi(X^\varepsilon)$, this inequality follow from lemma 7 by Bobkov-Götze's criterion [1], Theorem 3.1.

4 Proofs

First of all, one suppose that b and σ are bounded . We will prove both theorems (CLT and MDP).

4.1 Proof of Theorem 2

One has to prove that

$$\lim_{\varepsilon \to 0} \mathbb{E}(\|Y^\varepsilon - Y^0\|_\alpha) = 0.$$

$$\begin{aligned} Y^\varepsilon(t) =& \frac{1}{\sqrt{\varepsilon}}(X^\varepsilon(t) - X^0(t)) \\ =& \frac{1}{\sqrt{\varepsilon}} \int_0^t \{b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)\} ds + \int_0^t \sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) dW_s \end{aligned}$$

and

$$\begin{aligned} Y^0(t) =& \int_0^t [\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0) Y^0(s) + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0) Y^0(s-\tau) ds + \int_0^t \sigma(s, X_s^0, X_{s-\tau}^0) dW_s \\ Y^\varepsilon(t) - Y^0(t) =& \int_0^t [\sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - \sigma(s, X_s^0, X_{s-\tau}^0)] dW_s \\ &+ \int_0^t [\frac{b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)}{\sqrt{\varepsilon}} \\ &- \{\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0) Y_s^0(s) + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0) Y_{s-\tau}^0(s)\}] ds \\ =& A_1^\varepsilon(t) + A_2^\varepsilon(t) + A_3^\varepsilon(t) \end{aligned} \tag{15}$$

where

$$A_1^\varepsilon(t) = \int_0^t [\sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - \sigma(s, X_s^0, X_{s-\tau}^0)]dW_s$$

$$A_2^\varepsilon(t) = \int_0^t [\frac{b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)}{\sqrt{\varepsilon}} - \{\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)Y_s^\varepsilon(s) + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0)Y_{s-\tau}^0(s))\}]ds$$

$$A_3^\varepsilon(t) = \int_0^t [\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)(Y^\varepsilon(s) - Y^0(s)) + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0)(Y^\varepsilon(s-\tau) - Y^0(s-\tau))ds$$

First, we shall estimate $A_1^\varepsilon(t)A_2^\varepsilon(t), A_3^\varepsilon(t)$.

$$\mathbb{E}|A_1^\varepsilon(t)|^p \leq \int_0^t L_\sigma \mathbb{E}(|X^\varepsilon(t) - X^0(t)|^p).$$

Thanks to proposition 6, we have for $p > 2$

$$\begin{cases} \mathbb{E}(|X^\varepsilon(t) - X^0(t)|^p) & \leq \varepsilon^{\frac{p}{2}} C(p, K, L_\sigma, L_b) \\ \mathbb{E}|A_1^\varepsilon(t)|^p & \leq \varepsilon^{\frac{p}{2}} C(p, K, L_\sigma, L_b). \end{cases} \tag{16}$$

Using Taylor formula, there exist a random field $\theta^\varepsilon(t)$ $(0,1)^d$−valued such that :

$$\begin{aligned} \frac{b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)}{\sqrt{\varepsilon}} =& \frac{\partial b}{\partial x}(s, X_s^0 + \theta^\varepsilon(s)(X_s^\varepsilon - X_s^0), X_{s-\tau}^0)Y_s^\varepsilon(s) \\ &+ \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0 + \theta^\varepsilon(s)(X_{s-\tau}^\varepsilon - X_{s-\tau}^0))Y_{s-\tau}^\varepsilon(s) \end{aligned}$$

Since $\frac{\partial b}{\partial x}$ and $\frac{\partial b}{\partial y}$ are uniformly continuous, we have :

$$\begin{aligned} &|\frac{\partial b}{\partial x}(s, X_s^0 + \theta^\varepsilon(s)(X_s^\varepsilon - X_s^0, X_{s-\tau}^0) - \frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)| \\ &\quad \leq L_{\partial_x b}(|X_s^0 + \theta^\varepsilon(s)(X_s^\varepsilon - X_s^0) - X_s^0| + |X_{s-\tau}^0 + \theta^\varepsilon(s)(X_{s-\tau}^\varepsilon - X_{s-\tau}^0) - X_{s-\tau}^0|) \\ &\quad \leq L_{\partial_x b}(|\theta^\varepsilon(s)(X_s^\varepsilon - X_s^0)| + |\theta^\varepsilon(s)(X_{s-\tau}^\varepsilon - X_{s-\tau}^0)|) \\ &\quad \leq L_{\partial_x b}(|X_s^\varepsilon - X_s^0| + |X_{s-\tau}^\varepsilon - X_{s-\tau}^0|) \end{aligned}$$

Then,

$$|A_2^\varepsilon(t)| \leq \int_0^t |(X_s^\varepsilon - X_s^0)Y_s^\varepsilon| ds \leq \sqrt{\varepsilon} C \int_0^t |Y_s^\varepsilon|^2 ds$$

Thus, we get for $p > 2$

$$\mathbb{E}(|A_2^\varepsilon(t)|)^2 \leq \varepsilon^{\frac{p}{2}} C \tag{17}$$

$$|A_3^\varepsilon(t)| \leq \int_0^t [L_3|Y_s^\varepsilon - Y_s^0| + L_4|Y_{s-\tau}^\varepsilon - Y_{s-\tau}^0|]ds \tag{18}$$

Using Gronwall's inequality, we get

$$\mathbb{E}(|Y_s^\varepsilon - Y_s^0|^p + |Y_{s-\tau}^\varepsilon - Y_{s-\tau}^0|^p) \leq \varepsilon^{\frac{p}{2}} C(p, K, L_1)$$

For $p > 2$, $t \in [0, T]$ by Burkholder-Gandy-Devis's inequality, we get

$$\begin{aligned}\mathbb{E}|A_1^\varepsilon(t) - A_1^\varepsilon(s)|^p \leq & C_p \mathbb{E}(\int_s^t [\sigma(u, X_u^\varepsilon, X_{u-\tau}^\varepsilon) - \sigma(u, X_u^0, X_{u-\tau}^0)]^2 du)^{\frac{p}{2}} \\ \leq & C(p, L_b)\mathbb{E}(\int_s^t |X_s^\varepsilon - X_s^0|^2 du)^{\frac{p}{2}}\end{aligned}$$

Thanks to proposition 6, one has

$$\mathbb{E}|A_1^\varepsilon(t) - A_1^\varepsilon(s)|^p \leq C(p, K, L_\sigma, L_b)\varepsilon^{\frac{p}{2}} |t - s|^{\frac{p-2}{2}} \tag{19}$$

$$\begin{aligned}|A_2^\varepsilon(t) - A_2^\varepsilon(s)|^p \leq & (\int_s^t |\frac{b(u, X_u^\varepsilon, X_{u-\tau}^\varepsilon) - b(u, X_u^0, X_{u-\tau}^0)}{\sqrt{\varepsilon}} \\ & - \{\frac{\partial b}{\partial x}(u, X_u^0, X_{u-\tau}^0) Y_u^\varepsilon(u) + \frac{\partial b}{\partial y}(u, X_u^0, X_{u-\tau}^0) Y_{u-\tau}^0(u))\}|du)^p \\ \mathbb{E}|A_2^\varepsilon(t) - A_2^\varepsilon(s)|^p \leq & \mathbb{E}(\int_s^t L_1 \frac{1}{\sqrt{\varepsilon}}(|X_u^\varepsilon - X_u^0| + |X_{u-\tau}^\varepsilon - X_{u-\tau}^0|)du + \int_s^t [L_3|Y_u^\varepsilon| + L_4|Y_{u-\tau}^\varepsilon|du)^p \\ \leq & C(p, K, L_1, L_2, L_3, L_4)|t - s|^{\frac{p-2}{2}} \qquad (20)\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{E}|A_3^\varepsilon(t) - A_3^\varepsilon(s)|^p \leq & C(p, L_3, L_4)(\int_s^t [|Y_u^\varepsilon - Y_u^0| + |Y_{u-\tau}^\varepsilon - Y_{u-\tau}^0|]du)^p \\ \leq & C(p, K, L_1, L_3, L_4)\varepsilon^{\frac{p}{2}} |t - s|^{\frac{p-2}{2}} \qquad (21)\end{aligned}$$

Thus, by (19), (20), (21) there exist a constant $\check{C}$ independent of ε such that :

$$\mathbb{E}|Y_t^\varepsilon - Y_t^0 - (Y_s^\varepsilon - Y_s^0)|^p \leq \check{C}\varepsilon^{\frac{p}{2}} |t - s|^{\frac{p-2}{2}}$$

Then, for all $\alpha \in (0, \frac{1}{2})$, one can choose $p \in (2, 3)$ such that :

$$\lim_{\varepsilon \to 0} \mathbb{E}\|Y^\varepsilon - Y^0\|_\alpha = 0.$$

It concludes the proof of Theorem 2(CLT).

□

4.2 Proof of theorem 3

$Y_t^\varepsilon = \dfrac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}}$, $t \geq 0$. So Y_t^ε satisfies :

$$\begin{aligned} dY_t^\varepsilon &= \frac{1}{\sqrt{\varepsilon}}(b(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon) - b(t, X_t^0, X_{t-\tau}^0))dt + \sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dW_t \quad t \in (0, +\infty) \\ Y_t^\varepsilon &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [-\tau, 0] \end{aligned}$$

Since X_t^ε close to X_t^0 and $X_{t-\tau}^\varepsilon$ close to $X_{t-\tau}^0$, we have :

$$dY_t^\varepsilon \simeq [\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0)Y_t^\varepsilon + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^\varepsilon)Y_{t-\tau}^\varepsilon]dt + \sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon)dW_t$$

where $\dfrac{\partial b}{\partial x} = (\dfrac{\partial b}{\partial x_i})_{1\leq i\leq d}$ and $\dfrac{\partial b}{\partial y} = (\dfrac{\partial b}{\partial y_j})_{1\leq j\leq d}$.
Y^ε should be close to Y^0 determined by (13). Then, by Schilder's theorem and the contraction principle , Z^ε obeys the LDP with the speed $h^2(\varepsilon)$ and the rate function $I_{MDP}(h)$ given by (14).

Hence by Theorem 3, it is enough to show that Z^ε is $h^2(\varepsilon)-$exponentially equivalent to Z^0. In other words, we have to prove

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|Z^\varepsilon - Z^0\|_\alpha > \delta) = -\infty \qquad \forall\ \ \delta > 0 \tag{22}$$

As we first prove the case for b and σ bounded, we need the following Proposition.
Proposition 10 : *b, σ are Lipschitz functions, there exist some constant $C(p, K, C_b)$ which depend on p, K, C_b such that*

$$\mathbb{E}(\|X^\varepsilon - X^0\|_\infty)^p \leq \varepsilon^{\frac{p}{2}} C(p, K, C_b) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0$$

Proof :

$$X_t^\varepsilon - X_t^0 = \int_0^t (b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0))ds + \sqrt{\varepsilon}\int_0^t \sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)dW_s$$

$$\begin{aligned} \|X^\varepsilon - X^0\|_\infty^p \leq &(\sup_{t\in[0,T]} |\int_0^t [b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)]|)^p \\ &+ \varepsilon^{\frac{p}{2}}(\sup_{t\in[0,T]} |\int_0^t \sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)dW_s|)^p \end{aligned}$$

Let $J_1^\varepsilon(t) = \displaystyle\int_0^t [b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)]ds$
and $J_2^\varepsilon(t) = \displaystyle\int_0^t \sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)dW_s$.

Since b is lipschitz continuous function, by Hölder's inequality :

$$\mathbb{E}(\|J_1^\varepsilon\|_\infty)^p \leq C_b^p(\int_0^t 1ds)^{\frac{p}{q}}\mathbb{E}\int_0^t(\|X^\varepsilon - X^0\|_\infty)^p$$

where $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned}\mathbb{E}(\|J_2^\varepsilon\|_\infty)^p \leq& \varepsilon^{\frac{p}{2}}\mathbb{E}|\int_0^t \sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)dW_s|^p \\ \leq& \varepsilon^{\frac{p}{2}}\mathbb{E}|\int_0^t \sigma^2(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)ds|^{\frac{p}{2}} \\ \leq& \varepsilon^{\frac{p}{2}}C(K,p)\end{aligned}$$

Therefore, there exists some constant $C(p, K, C_b)$ such that

$$\mathbb{E}(\|X^\varepsilon - X^0\|_\infty)^p \leq C(p, K, C_b)(\mathbb{E}\int_0^T(\|X^\varepsilon - X^0\|_\infty)^p dt + \varepsilon^{\frac{p}{2}})$$

By Gronwall's lemma, one obtain

$$\mathbb{E}(\|X^\varepsilon - X^0\|_\infty)^p \leq \varepsilon^{\frac{p}{2}}C(p, K, C_b)e^{C(p,K,C_b)}$$

Recall

$$\begin{aligned}Y_t^\varepsilon =& \frac{X_t^\varepsilon - X_t^0}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}}\int_0^t[b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)]ds + \int_0^t[\sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)]dW_s \\ Y_t^0 =& \int_0^t[\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)Z_s^0 + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0)Z_{s-\tau}^0]ds \\ &+ \int_0^t[\sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon)]dW_s \\ d(Y_t^\varepsilon - Y_t^0) =& [\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0)(Z_t^\varepsilon - Z_t^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)(Z_{t-\tau}^\varepsilon - Z_{t-\tau}^0)]dt \\ &+ [\sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon) - \sigma(t, X_t^0, X_{t-\tau}^0)]dW_t \\ &+ \frac{1}{\sqrt{\varepsilon}}[(b(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon) - b(t, X_t^0, X_{t-\tau}^0)) \\ &- (\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0)(X_t^\varepsilon - X_t^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)(X_{t-\tau}^\varepsilon - X_t^0))]dt \\ =& [\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0)(X_t^\varepsilon - X_t^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)(X_{t-\tau}^\varepsilon - X_{t-\tau}^0)]dt + dU_t^\varepsilon\end{aligned}$$

$$
\begin{aligned}
U_t^\varepsilon =& \int_0^t \sigma(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - \sigma(s, X_s^0, X_{s-\tau}^0)]dW_s \\
&+ \frac{1}{\sqrt{\varepsilon}} \int_0^t [(b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0)) \\
&- (\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)(X_s^\varepsilon - X_s^0) + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0)(X_{s-\tau}^\varepsilon - X_{s-\tau}^0))]ds \\
Y_t^\varepsilon - Y_t^0 =& U_t^\varepsilon + \int_0^t [\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)]J(s,t)U_t^\varepsilon
\end{aligned}
$$

where $J(s,s) = Id$ and
$\frac{db}{dt} = (\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0))J(s,t)U_t^\varepsilon(s)ds$ for $0 \le s \le t$

Since

$$
|\langle \frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0), x + y\rangle| \le L(|x|^2 + |y|^2)
$$

One has

$$
|J(s,t)u| \le C(p, K, C_b)|u| \qquad \forall y \in \mathbb{R}^d
$$

As

$$
\|\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)\|_\infty = \sup_{x,y,v \in \mathbb{R}^d, \|u\| \le 1} \|[\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)](u)\|
$$

One has, for all $t \in [0, T]$,

$$
\begin{aligned}
|Y_t^\varepsilon - Y_t^0| &\le |U_t^\varepsilon| + \|\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)\|_\infty C(p, K, C_b) \int_0^t |U_s^\varepsilon| ds \\
&\le (1 + \|\frac{\partial b}{\partial x}(t, X_t^0, X_{t-\tau}^0) + \frac{\partial b}{\partial y}(t, X_t^0, X_{t-\tau}^0)\|_\infty \frac{C(p, K, C_b)}{L})\|U^\varepsilon\|_\infty \qquad (23)
\end{aligned}
$$

One has to prove

Proposition 11 : *For all $r > 0$*

$$
\limsup_{\varepsilon \to 0} h^{-2}(\varepsilon) \log P(\frac{\|U^\varepsilon\|_\infty}{h(\varepsilon)} > r) = -\infty
$$

Proof

$$\begin{aligned}|U_t^\varepsilon| \leq &|\int_0^t (\sigma(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-\sigma(s,X_s^0,X_{s-\tau}^0))dW_s| \\ &+\int_0^t \frac{1}{\sqrt{\varepsilon}}|(b(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-b(s,X_s^0,X_{s-\tau}^0)) \\ &-(\frac{\partial b}{\partial x}(s,X_s^0,X_{s-\tau}^0)(X_s^\varepsilon-X_s^0)+\frac{\partial b}{\partial y}(s,X_s^0,X_{s-\tau}^0)(X_{s-\tau}^\varepsilon-X_{s-\tau}^0))|ds \\ :=&|M_t^\varepsilon|+\int_0^t \frac{1}{\sqrt{\varepsilon}}|(b(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-b(s,X_s^0,X_{s-\tau}^0))(X_s^\varepsilon-X_s^0)|ds\end{aligned}$$

where M_t^ε is a continuous martingale with

$$\langle M^\varepsilon\rangle_t=\int_0^t (\sigma(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-\sigma(s,X_s^0,X_{s-\tau}^0))(\sigma(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-\sigma(s,X_s^0,X_{s-\tau}^0))^*ds$$

For all $\eta>0$, one has :

$$\begin{aligned}P(\sup_{0\leq t\leq T}|M_t^\varepsilon|\geq \frac{rh(\varepsilon)}{2}) &\leq P(\sup_{0\leq t\leq T}|M_t^\varepsilon|\geq \frac{rh(\varepsilon)}{2};\langle M^\varepsilon\rangle_t\leq\eta)+P(\langle M^\varepsilon\rangle_T\geq\eta) \\ &\leq 2\exp(-\frac{\frac{(rh(\varepsilon))^2}{2^2}}{2\eta})+\mathbb{P}(\langle M^\varepsilon\rangle_T\geq\eta) \\ &\leq 2\exp(-\frac{(rh(\varepsilon))^2}{2\eta})+P(\langle M^\varepsilon\rangle_T\geq\eta)\end{aligned}$$

Since,

$$\begin{aligned}\langle M^\varepsilon\rangle_T &=\int_0^T (\sigma(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-\sigma(s,X_s^0,X_{s-\tau}^0))(\sigma(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-\sigma(s,X_s^0,X_{s-\tau}^0))^*ds \\ &\leq L\int_0^T |X_s^\varepsilon-X_s^0|ds=\varepsilon L\int_0^T |Y_s^\varepsilon|^2ds\end{aligned}$$

for some L. for ε small enough, by proposition 1, one has :

$$\mathbb{E}(\int_0^T |Y_s^\varepsilon|^2ds)^{\frac{1}{2}}\leq\frac{1}{2}(\frac{\eta}{\varepsilon L})^{\frac{1}{2}}$$

Using lemma 8, one has :

$$\begin{aligned}\mathbb{P}(\langle M^\varepsilon\rangle_T \geq \eta) &\leq \mathbb{P}(\int_0^T |Y_s^\varepsilon|^2 ds \geq \frac{\eta}{\varepsilon L})\\ &\leq P((\int_0^T |Y_s^\varepsilon|^2 ds)^{\frac{1}{2}} - \mathbb{E}(\int_0^T |Y_s^\varepsilon|^2 ds)^{\frac{1}{2}} \geq (\frac{\eta}{\varepsilon L})^{\frac{1}{2}} - \frac{1}{2}(\frac{\eta}{\varepsilon L})^{\frac{1}{2}})\\ &\leq P((\int_0^T |Y_s^\varepsilon|^2 ds)^{\frac{1}{2}} - \mathbb{E}(\int_0^T |Y_s^\varepsilon|^2 ds)^{\frac{1}{2}} \geq \frac{1}{2}(\frac{\eta}{\varepsilon L})^{\frac{1}{2}})\\ &\leq \exp\{-\frac{\eta}{8\varepsilon C_T L}\}\end{aligned}$$

Notice that $\varepsilon h^2(\varepsilon) \longrightarrow 0$ and $\eta > 0$ is taken arbitrary, one get :

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon) \log P(\sup_{0\leq t\leq 1} |M_t^\varepsilon| \geq \frac{rh(\varepsilon)}{2}) = -\infty \tag{24}$$

Since $b \in C^1$ and $\frac{\partial b}{\partial x}, \frac{\partial b}{\partial x}$ are uniformly continuous, for all $\eta > 0$ there exist some $\delta > 0$ such that :

$$|b(s,x,y) - b(s,u,y) - \frac{\partial b}{\partial x}(s,x,y)(x-u)| \leq |x-u|$$

as $|X_t^\varepsilon - X_t^0| \leq \delta$

$$|b(s,x,y) - b(s,x,v) - \frac{\partial b}{\partial y}(s,x,y)(y-v)| \leq |y-v|$$

as $|X_{t-\tau}^\varepsilon - X_{t-\tau}^0| \leq \delta$

Hence,

$$\begin{aligned}P(\int_0^T \frac{1}{\sqrt{\varepsilon}}|b(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon) - b(s,X_s^0,X_{s-\tau}^0) - \frac{\partial b}{\partial x}(s,X_s^0,X_{s-\tau}^0)(X_s^\varepsilon - X_s^0)&\\ -(\frac{\partial b}{\partial y}(s,X_s^0,X_{s-\tau}^0)(X_{s-\tau}^\varepsilon - X_{s-\tau}^0)|ds \geq \frac{rh(\varepsilon)}{2})&\\ \leq P(\|X^\varepsilon - X^0\|_\infty \geq \delta) + P(\int_0^T |Z_t^\varepsilon| dt \geq \frac{rh(\varepsilon)}{4\eta})&\\ + P(\int_0^T |Z_{t-\tau}^\varepsilon| dt \geq \frac{rh(\varepsilon)}{4\eta})&\end{aligned}$$

By Freidlin-Wentzell Theorem :

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(\|X^\varepsilon - X^0\|_\infty \geq \delta) = 0$$

and

$$\limsup_{\varepsilon\to 0} \varepsilon h^{-2}(\varepsilon)\log P(\|X^\varepsilon - X^0\|_\infty \geq \delta) = 0$$

Since ε is sufficiently small, $\mathbb{E}(\int_0^T |Z_t^\varepsilon|^2 dt)^{\frac{1}{2}} \leq \frac{rh(\varepsilon)}{4\eta\sqrt{T}}$ by lemma 7.

Using Cauchy-Schwartz inequality and lemma 8, one get

$$\begin{aligned} P(\int_0^T |Z_s^\varepsilon| dt \geq \frac{rh(\varepsilon)}{2\eta}) &\leq P((\int_0^T |Z_s^\varepsilon| dt)^{\frac{1}{2}}) \geq \frac{rh(\varepsilon)}{2\eta}) \\ &\leq P((\int_0^T |Z_s^\varepsilon| dt)^{\frac{1}{2}} - \mathbb{E}(\int_0^T |Z_s^\varepsilon| dt)^{\frac{1}{2}}) \geq \frac{rh(\varepsilon)}{4\eta\sqrt{T}}) \\ &\leq \exp\{-\frac{(rh(\varepsilon))^2}{32\eta^2 C_T}\} \end{aligned}$$

Analogously,

$$P(\int_0^T |Z_{s-\tau}^\varepsilon| dt \geq \frac{rh(\varepsilon)}{2\eta}) \leq \exp\{-\frac{(rh(\varepsilon))^2}{32\eta^2 C_T}\}$$

Since η is taken arbitrary, one has :

$$\begin{aligned} \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\int_0^T \frac{1}{\sqrt{\varepsilon}} |b(s, X_s^\varepsilon, X_{s-\tau}^\varepsilon) - b(s, X_s^0, X_{s-\tau}^0) - \frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)(X_s^\varepsilon - X_s^0)) \\ - (\frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0)(X_{s-\tau}^\varepsilon - X_{s-\tau}^0))| ds \geq \frac{rh(\varepsilon)}{2}) = -\infty \end{aligned} \tag{25}$$

One complete the proof of proposition 11 by (23), (24) and (25).

Next, one has to prove in Hölder norm.

Lemma 12 : *For all $\delta > 0$,*

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|Z^\varepsilon - Z^0\|_\alpha > \delta) = -\infty$$

Proof :

$$I_3^\varepsilon(t) = \int_0^t [\frac{\partial b}{\partial x}(s, X_s^0, X_{s-\tau}^0)(Y_s^\varepsilon - Y_s^0) + \frac{\partial b}{\partial y}(s, X_s^0, X_{s-\tau}^0)(Y_{s-\tau}^\varepsilon - Y_{s-\tau}^0)] ds$$

Using Hölder's inequality, one has

$$\begin{aligned}|I_3^\varepsilon(t)-I_3^\varepsilon(s)| \leq& (\int_s^t |\frac{\partial b}{\partial x}(s,X_s^0,X_{s-\tau}^0)|^q du)^{\frac{1}{q}}(\int_s^t |Y_s^\varepsilon - Y_s^0|^p du)^{\frac{1}{p}} \\ &+ \leq (\int_s^t |\frac{\partial b}{\partial y}(s,X_s^0,X_{s-\tau}^0)|^q du)^{\frac{1}{q}}(\int_s^t |Y_{s-\tau}^\varepsilon - Y_{s-\tau}^0|^p du)^{\frac{1}{p}} \\ \leq& C^q_{\partial_x,\partial_y}|t-s|^{\frac{1}{q}}[(\int_s^t |Y_u^\varepsilon|^p du)^{\frac{1}{p}} + (\int_s^t |Y_{u-\tau}^\varepsilon|^p du)^{\frac{1}{p}}]\end{aligned}$$

Choose $q \in (2;3), \alpha = \dfrac{1}{q}$, one get

$$\|I_3^\varepsilon\|_\alpha \leq C_{\partial_x,\partial_y}(\int_s^t (1+u)^\alpha (\|Y^\varepsilon - Y^0\|_\alpha)^p du)^{\frac{1}{p}}$$

Hence, for $t \in [0,1]$, one has

$$\|Y_u^\varepsilon - Y_u^0\|_\alpha \leq C(p,\partial_x b,\partial_y b)[\|I_1^\varepsilon\|_\alpha^p + \|I_2^\varepsilon\|_\alpha^p + \int_s^t (\|Y_u^\varepsilon - Y_u^0\|_\alpha)^p du]^{\frac{1}{p}}$$

Applying Gronwall's lemma to $g(t) = (\|Y^\varepsilon - Y^0\|_\alpha)^p$, one has :

$$(\|Y^\varepsilon - Y^0\|_\alpha)^p \leq C(p, C_{\partial_x,\partial_y})(\|I_1^\varepsilon\|_\alpha + \|I_2^\varepsilon\|_\alpha)^p e^{C(p,T,C_{\partial_x b,\partial_y b})}$$

It suffices to show that for $\delta > 0$,

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\frac{\|I_i^\varepsilon\|_\alpha}{h(\varepsilon)} \leq \delta) = -\infty \qquad i = 1,2$$

For $\varepsilon > 0, \eta > 0$,

$$P(\|I_1^\varepsilon\|_\alpha \geq h(\varepsilon)\delta) \leq P(\|I_1^\varepsilon\|_\alpha \geq h(\varepsilon)\delta, \|X^\varepsilon - X^0\|_\infty < \eta) + P(\|X^\varepsilon - X^0\|_\infty \geq \eta)$$

Since

$$|(\sigma(s,x,y)-\sigma(s,u,v))(\sigma(s,x,y)-\sigma(s,u,v))^*| \leq L(|x-u|^2 + |y-v|^2)$$

then, for each $s \geq 0, 0 < \alpha < \dfrac{1}{2}$ and $h(\varepsilon)\delta \geq \eta d L$,

$$P(\|I_1^\varepsilon\|_\alpha \geq h(\varepsilon)\delta, \|X^\varepsilon - X^0\|_\infty < \eta) \leq 2d\exp(-\frac{(h(\varepsilon)\delta)^2}{2\eta^2 l d^2})$$

Since X^ε satisfy an LDP on $C_\psi^{\alpha,0}([0,m]:\mathbb{R}^d)$

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\log P(\|X^\varepsilon - X^0\|_\infty \geq \eta) &\leq \limsup_{\varepsilon\to 0}\varepsilon\log P(\|X^\varepsilon - X^0\|_\alpha \geq \eta)\\ &\leq -\inf\{\tilde{I}^*(f);\|f-X^0\|_\alpha\geq\eta\}\end{aligned}$$

As $\tilde{I}^*$ the rate function for the LDP is a good rate function ; it admits a compact set. One reach $\inf\{\tilde{I}^*(f);\|f-X^0\|_\alpha\geq\eta\}$ to some function f_0. $\tilde{I}^*(f)=0$ if $f=X^0$, we may conclude :

$$\inf\{\tilde{I}^*(f);\|f-X^0\|_\alpha\geq\eta\}<0$$

Since $h(\varepsilon)\to\infty$ and $\sqrt{\varepsilon}h(\varepsilon)\longrightarrow 0$, when $\varepsilon\longrightarrow 0$, one has :

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|X^\varepsilon - X^0\|_\alpha\geq\eta)=-\infty$$

As we take $\eta>0$ arbitrary, one get :

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\frac{\|I_1^\varepsilon\|_\alpha}{h(\varepsilon)}\leq\delta)=-\infty$$

For

$$\begin{aligned}I_2^\varepsilon(t)=\int_0^t[\frac{1}{\sqrt{\varepsilon}}(b(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)-b(s,X_s^0,X_{s-\tau}^0))-(\frac{\partial b}{\partial x}(s,X_s^0,X_{s-\tau}^0)Y_s^\varepsilon)\\ -(\frac{\partial b}{\partial y}(s,X_s^0,X_{s-\tau}^0)Y_{s-\tau}^\varepsilon)]ds\end{aligned}$$

Similarly as above, (in the proof of CLT) , one has :

$$\|I_2^\varepsilon(t)\|_\alpha\leq C_{\partial_x b,\partial_y b}\int_0^1\frac{(\|X^\varepsilon - X^0\|_\alpha)^2}{\sqrt{\varepsilon}}dt$$

Following the same argument as in the proof of proposition 1

$$\|X^\varepsilon - X^0\|_\alpha\leq\|\tilde{I}_2^\varepsilon\|_\alpha$$

where

$$\tilde{I}_2^\varepsilon(t)=\sqrt{\varepsilon}\int_0^t\sigma(s,X_s^\varepsilon,X_{s-\tau}^\varepsilon)dW_s$$

Applying lemma 8 with $|\sigma(s,x,y)|\leq K$. For each $\eta>0$, one obtain for all ε sufficiently small such that $h(\varepsilon)\delta\geq\sqrt{\varepsilon}KL$

$$P(\|\tilde{I}_2^\varepsilon\|_\alpha\geq h(\varepsilon)\delta,|X^\varepsilon|<|X^0|+\eta)\leq 2d\exp(\frac{(h(\varepsilon)\delta)^2}{\varepsilon K^2 l(1+|X^0|+\eta)^2})$$

Then,

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|X^\varepsilon\|_\alpha \geq \|X^0\|_\alpha+\eta) \leq \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|X^\varepsilon - X^0\|_\alpha \geq \eta)$$

So, one has :

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\frac{\|I_2^\varepsilon\|_\alpha}{h(\varepsilon)} \geq \delta) \leq \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\frac{\|\tilde{I}_2^\varepsilon\|_\alpha^2}{h(\varepsilon)} \geq \frac{\sqrt{\varepsilon}h(\varepsilon)\delta}{C(\alpha,K,C_b)})$$

$$\begin{aligned}
&\leq \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\frac{\|\tilde{I}_2^\varepsilon\|_\alpha^2}{h(\varepsilon)} \geq \frac{\sqrt{\varepsilon}h(\varepsilon)\delta}{C(\alpha,K,C_b)}, \|X^\varepsilon\|_\alpha \geq \|X^0\|_\alpha+\eta)\\
&\quad + \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|X^\varepsilon\|_\alpha \leq \|X^0\|_\infty+\eta)\\
&\leq (\limsup_{\varepsilon\to 0} \frac{-\delta}{\sqrt{\varepsilon}h(\varepsilon)C(\alpha,K,C_b)K^2L(1+\|X^0\|_\alpha+\eta)^2})\\
&\quad \vee (\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log P(\|X^\varepsilon\|_\alpha \leq \|X^0\|_\alpha+\eta)\\
&= -\infty
\end{aligned}$$

The proof of the MDP in the case of bounded coefficients is complete.

□

4.3 σ and b unbounded

For $R>0$, define $m_R := \sup\{|b(t,x,y)|, |\sigma(t,x,y)|; t\in[0,m], |x|\leq R, |y|\leq R\}$ and $b_i^R := (-m_R-1)\vee b_i \wedge (m_R+1)$, $\sigma_{i,j}^R := (m_R+1)$, $1\leq i,j\leq 2d$.
Set $b_R := (b_i^R)_{1\leq i\leq 2d}$ and $\sigma_R := (\sigma_{i,j}^R)_{1\leq i,j\leq 2d}$.
Then, $b_R(t,x,y) := b(t,x,y)$, $\sigma_R(t,x,y) := \sigma(t,x,y)$ for $t\in[0,m]$, $|x|\leq R$ and $|y|\leq R$. Moreover, b_R and σ_R satisfy Lipschitz condition.

Recall two results found in Mohammed and Zhang [13].

Proposition 13 :(Proposition 3.5 in [13]) *Assume*

$$|b(t,x,y)| \leq C(1+|x|+|y|)$$

$$|\sigma(t,x,y)| \leq C(1+|x|+|y|)$$

for all $x,y\in\mathbb{R}^d$
Then, for all $m\geq 1$,

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{-\tau\leq t\leq m})(|X_t^\varepsilon|>R) = -\infty$$

.
where X^ε is the solution to

$$\begin{cases} dX_t^\varepsilon = & b(t,X_t^\varepsilon,X_{t-\tau}^\varepsilon)dt+\sqrt{\varepsilon}\sigma(t,X_t^\varepsilon,X_{t-\tau}^\varepsilon)dW_t & t\in(0,\infty) \\ X_t^\varepsilon = & \psi(t) & t\in[-\tau,0] \end{cases}$$

Let $X_t^{\varepsilon,R}$ solution to the sdde

$$\begin{cases} dX_t^{\varepsilon,R} = & b_R(t,X_t^{\varepsilon,R},X_{t-\tau}^{\varepsilon,R})dt+\sqrt{\varepsilon}\sigma_R(t,X_t^{\varepsilon,R},X_{t-\tau}^{\varepsilon,R})dW_t & t\in(0,\infty) \\ X_t^{\varepsilon,R} = & \psi(t) & t\in[-\tau,0] \end{cases}$$

Proposition 14 : (Proposition 3.6 in [13]) *Assume*

$$|b(t,x,y)|\le C(1+|x|+|y|)$$

$$|\sigma(t,x,y)|\le C(1+|x|+|y|)$$

for all $x\in\mathbb{R}^{2d}$

Fix $m\ge 1$, *then,*

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{-\tau\le t\le m})(|X_t^\varepsilon-X_t^{\varepsilon,R}|>\delta)=-\infty$$

Now for any $R>0$ large enough, $\sigma_R(t,x,y)=\sigma(t,x,y)$ and $b_R(t,x,y)=b(t,x,y)$ for $x,y\in\mathbb{R}^d$ and $t\in(-\tau,m)$ with $|x|\le R$ and $|y|\le R$, such that σ_R is globally Lipschitzian function, and b_R is C^1 with $Db_R=\partial_x b_R+\partial_y b_R$ uniformly continuous.

Consider the solution $Y_t^{\varepsilon,R}$ of the corresponding sdde

$$\begin{aligned} dY_t^{\varepsilon,R} &= \frac{1}{\sqrt{\varepsilon}}(b_R(t,X_t^\varepsilon,X_{t-\tau}^\varepsilon)-b_R(t,X_t^0,X_{t-\tau}^0))dt+\sigma_R(t,X_t^\varepsilon,X_{t-\tau}^\varepsilon)dW_t \quad t\in(0,+\infty) \\ Y_t^{\varepsilon,R} &= 0 \quad t\in[-\tau,0] \end{aligned}$$

We have , by Proposition 14 :

$$\begin{aligned} \limsup_{\varepsilon\to 0}\varepsilon\log P(X_t^\varepsilon\ne X_t^{\varepsilon,R}\ \ for\ \ some\ \ t\in[0,m]) &\le \limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{-\tau\le t\le m}|Y^\varepsilon(t)|>R) \\ &\le -\infty \end{aligned}$$

Then, following the same argument as in the previous case (b and σ bounded) , we complete the proof of both results (Theorem 2 and Theorem 3).

□

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