Journal de Mathématiques,
Mathématiques & Applications
Fondamentales et Informatique
JMMAFI 2014, Vol.**1**,pp 17-32

# A large deviations principle for random evolution delay equations in hölder space

S.H. RANDRIAMANIRISOA[1] , T.J. RABEHERIMANANA[2]

Département de Mathématiques et Informatique, B.P. 906, Ankatso 101, Antananarivo, Madagascar

[1] *hasinasran@gmail.com*, [2] *rabeherimanana.toussaint@gmx.fr*

**Résumé:** Dans cet article, nous étudions un principe de grandes déviations concernant les équations d' évolutions aléatoires avec délai en norme höldérienne.
**Mots-clés:** principe de grandes déviations, équations d'évolution aléatoire avec retard, espace höldérien
**Abstract:** In this paper, we develop a large deviations principle for random evolution delay equations driven by small multiplicative white noise in hölderian space.
**Keywords:** large deviations principle,random evolution delay equation, hölder space.

## Introduction

For the last decades, Large deviations took the interests of many author. The results found with large deviations principle have important applications to many areas including economics and finance field(see [7]), signal processing, statistical tests(see [3]), information theory, communication networks, risk-sensitive control and statistical mechanics(see [14], [18]). First rigourous results in large deviations have been found by H. Cramer [4], he has used it to simulate insurance operations.Then, S.R.S. Varadhan [18] developed fundamental work about large deviations principle in 1967. After, large deviations theory has been studied by more and more mathematicians, we can note the works of Schilder [17],Sanov [15] ,and Freidlin and Wentzell[7]which show large deviations principle basis tool.

Therefore, there is yet a little published papers about large deviations for system with memory. M Scheutzow [16] is the precursor of study of large deviations for stochastic system with memory, he worked in the context of additive white noise.

In this paper, we observe the case of large deviations for random evolution delay equations in strong topology. Our approach is similar to that in Mellouk [10], with taking account induction argument in to handle the delay.

This paper is organized as follow: basic setting and notation are given in the first section. In section 2, we have some definitions and general results on the Large deviations principle(LDP).Then, the main result concerning the LDP for the solution of the stochastic differential delay equation (3.1) is stated in section 3. Section 4 is devoted to a general approximate contraction principle that meets our needs. In the last section(section 5), we prove that this version of the contraction principle can be applied to prove the main result.

Throughout this paper, we will use the following notational convention: for the proof, constants appearing are denoted by C, eventhough they may change from one line to the next one.

# 1. Basic setting and notation

Let $W_t := (W_t^1, W_t^2, ..., W_t^l)$ denote a standard l-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with $W_0 = 0$ , $\Omega = \mathbb{C}_0([0,m], \mathrm{R}^l)$ equipped with the usual topology of uniform convergence defined by the norm $\|f\|_\infty = \sup_{0\leq t\leq m}|f(t)|$ . For $0 < \alpha < \frac{1}{2}$ we define the $\alpha$-Hölder space $\mathbb{C}_\psi^\alpha([-1,m], \mathrm{R}^d)$ as the space of continuous functions f such that

$$\|f\|_\alpha = \sup_{s,t\in[0,m]} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty.$$

Define the Hölderian modulus of continuity of f by

$$\omega_\alpha(f,\delta) = \sup_{0<|t-s|\leq\delta; s,t\in[0,m]} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

Let $b = (b_1, b_2, ..., b_d) : \mathrm{R}^+ \times \mathrm{R}^d \times \mathrm{R}^d \times \mathrm{R}^d \longrightarrow \mathrm{R}^d$ and $\sigma = (\sigma_{ij})_{i=1,...,d,j=1,...,l} : \mathrm{R}^+ \times \mathrm{R}^d \times \mathrm{R}^d \times \mathrm{R}^d \longrightarrow \mathrm{R}^d \times \mathrm{R}^l$ be Borel measurable functions.

Let $\tau > 0$ be a fixed delay, and $\psi$ be a given continuous function on $[-\tau, 0]$.

Consider the following differential delay equation (dde):

$$\text{(I)} \begin{cases} dX(t) = dX_t = b(t, X_t, X_{t-\tau}, Y_t)dt & \text{t} \in (0,\infty) \\ X(t) = \psi(t) & \text{t} \in [-\tau, 0] \end{cases}$$

and the associated perturbed stochastic differential delay equation (sdde)

$$\text{(II)} \begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon, Y(t))dt + \varepsilon^{\frac{1}{2}}\sigma(t, X_t^\varepsilon, X_{t-\tau}^\varepsilon, Z_t)dW_t & t \in (0,\infty) \\ X_t^\varepsilon = \psi(t) & \text{t} \in [-\tau, 0] \end{cases}$$

Throughout this paper, we will assume, without loss of generality, that the delay $\tau$ is equal to 1.

For $0 < \alpha < \frac{1}{2}$, let $C_0^\alpha([-1,m], \mathrm{R}^d)$ be the separable space of $\alpha$-Hölder continuous functions $g : [0,m] \longrightarrow \mathrm{R}^l$ with $g(0) = 0$.

For $0 < \alpha < \frac{1}{2}$, denote by $C_\psi^\alpha([-1,m], \mathrm{R}^d)$ be the set of all $\alpha$-Hölder continuous functions $f : [-1,m] \longrightarrow \mathrm{R}^d$ such that $f(t) = \psi(t)$ for all $t \in [-1,0]$.

The goal of this paper is to derive an LDP in $C_\psi^\alpha([-1,m], \mathrm{R}^d)$ for random evolution delay equations.

# 2. Definitions and general results

Let E be a topological space and $\mathcal{F}$ its Borel $\sigma$-field, and let $\{P_\varepsilon, \varepsilon > 0\}$ be a family of probability measures on $(E, \mathcal{F})$. We begin by giving several definitions.

***Definition 2.1:*** *A function $I : E \longrightarrow [0,\infty]$ is said to be a rate function if it is lower semicontinuous(lsc) i.e. $\forall x_n \to x, I(x) \leq \underline{\lim} I(x_n)$ Furthermore,* *if, for each $a < \infty$, $\Gamma_a = \{x \in E; I(x) \leq a\}$ is compact. We will say that I is a good rate function.*

***Definition 2.2:*** *For some function I, the probabilities $\{P_\varepsilon\}_{\varepsilon>0}$ satisfy a large deviations principle if the following hold:*

1. *I is a good rate function .*
2. *(lower bound)for every open subset G of E*

$$\liminf_{\varepsilon\to 0} \varepsilon \ln P_\varepsilon(G) \geq -I(G).$$

3. *(upper bound) For every closed subset F of E*

$$\limsup_{\varepsilon\to 0} \varepsilon \ln P_\varepsilon(F) \leq -I(F).$$

It is well known that $C^\alpha_\psi([-1,m];\mathrm{R}^d)$ is not separable but its closed subspace defined by $C^{\alpha,0}_\psi([-1,m];\mathrm{R}^d) = \{f \in C^\alpha_\psi([-1,m];\mathrm{R}^d); \lim_{\delta\to 0}\omega_\alpha(f,\delta) = 0\}$ is separable.

Both $C^\alpha_\psi([-1,m];\mathrm{R}^d)$ and $C^{\alpha,0}_\psi([-1,m];\mathrm{R}^d)$ are Banach spaces for the norm $\|.\|_\alpha$ and $\|.\|_\infty \leq \|.\|_\alpha$. It is well known that $P(\|W\|_\alpha < \infty) = 1$ for $0 < \alpha < \frac{1}{2}$. We shall need the following version of the Arzelà-Ascoli theorem .

***Theorem 2.3:*** *A set $A \subset C^{\alpha,0}_\psi([-1,m];\mathrm{R}^d)$ has compact closure in $C^{\alpha,0}_\psi([-1,m];\mathrm{R}^d)$ if and only if the following two conditions hold :*

$$\sup_{f\in A}\|f\|_\alpha < \infty$$

and

$$\lim_{\delta\downarrow 0}\sup_{f\in A}\omega_\alpha(f,\delta) = 0.$$

Let $\mathcal{H}$ denote the Cameron-Martin space i.e

$$\mathcal{H} = \{h(t) = \int_0^t \dot{h}_s ds : [0,m] \to \mathbb{R}^l : \int_0^m |\dot{h}(s)|^2 ds < +\infty\}.$$

This is an Hilbert space endowed with the inner product defined by :

$$\langle f, g\rangle_{\mathcal{H}} = \int_0^m \dot{f}_s \dot{g}_s ds. \tag{2.1}$$

If $g \in C^\alpha_0([0,m];\mathrm{R}^l)$ is absolutely continuous, that is, $g : [0,m] \longrightarrow \mathrm{R}^l$ with $g(0) = 0$, set
$e(g) = \int_0^m |\dot{g}(s)|^2 ds$ . Otherwise, define $e(g) = +\infty$. (2.2)

***Theorem 2.4:*** *A good rate function $I(f) = \inf\{\frac{1}{2}e(g); F(g) = f\}, f \in C^{\alpha,0}_\psi([-1,m];\mathrm{R}^d)$ and F(g) is a solution to the sdde (II).*

***Theorem 2.5:*** *The probability measures induced by $\sqrt{\varepsilon}W$ on $C^{\alpha,0}_0([0,m];\mathrm{R}^d)$ equipped with the norm $\|.\|_\infty$ satisfy the LDP with the good rate function I(.) defined by $I(g) = \frac{1}{2}\|g\|^2_{\mathcal{H}}$.* (2.3)

It extends the LDP proved by P. Baldi and al[1] which extends the classical Schilder theorem. We now state the Hölder version of classical exponential inequality for stochastic integrals, which is crucial in proving the exponential approximations.

***Lemma 2.6:*** *Let* $f : [0,1] \times \Omega \longrightarrow \mathrm{R}^l \times \mathrm{R}^d$ *and* $g : [0,1] \times \Omega \longrightarrow \mathrm{R}^l$ *be bounded* $(\mathcal{F}_t)$*-progressively measurable functions, and set*
$U(t) = \int_0^t f(s)dW_s + \int_0^t g(s)ds; 0 \le t \le 1.$
*Define* $A = \sup_{t,\omega} tr(f(t,\omega)f^T(t,\omega))$ *and* $B = \sup_{t,\omega} g(t,\omega)$. *Then, for every* $s \ge 0, T \ge 0$, $0 < \alpha < \frac{1}{2}$ *and* $r > lBT^{1-\alpha}$

$$P(\sup_{s \le t \le s+T} \frac{|U(t) - U(s)|}{|t-s|^\alpha}) \le 2l \exp[-\frac{(r - lBT^{1-\alpha})^2}{2Al^2T^{1-2\alpha}}] \tag{2.4}$$

# 3. The main result

In this section, we give conditions under which the solution of the sdde defined by

$$X_t^\varepsilon = x + \int_0^t b(s, X_s^\varepsilon, X_{s-1}^\varepsilon, Y_s)ds + \varepsilon^{\frac{1}{2}} \int_0^t \sigma(s, X_s^\varepsilon, X_{s-1}^\varepsilon, Z_s)dW_s \tag{3.1}$$

(where $x \in \mathrm{R}^d$, W is a standard Brownian motion) satisfies an LDP in any Hölder norm with the exponent $\alpha < \frac{1}{2}$.

Let $Y = \{Y(t), 0 \le t \le m\}$ be a $\mathrm{R}^d$-valued process which is $\{\mathcal{F}_t\}$-progressively measurable. In order to make explicit the LDP rate function for the law of (I) in the $\alpha$-Hölder topology , we suppose that Y is a random variable with values in $L^{\frac{1}{1-\alpha}}([0,m], \mathrm{R}^d)$ . Let $Z = \{Z(t); 0 \le t \le m\}$ be an $\mathcal{F}_t$-progressively measurable process taking values in $\mathrm{R}^d$. We assume that suppZ is a compact subset in $C^{\alpha,0}([0,m], \mathrm{R}^d)$, and that $(Y, Z)$ and W are independent.

From now on, we suppose that the coefficients $\sigma$ and b satisfy the following conditions:

$(H_0)$ $b = (b_1, b_2, ..., b_d) : \mathrm{R}^+ \times \mathrm{R}^d \times \mathrm{R}^d \times \mathrm{R}^d \longrightarrow \mathrm{R}^d$ and
$\sigma = (\sigma_{ij})_{i=1,...,d, j=1,...,l} : \mathrm{R}^+ \times \mathrm{R}^d \times \mathrm{R}^d \times \mathrm{R}^d \longrightarrow \mathrm{R}^d \times \mathrm{R}^l$ are Borel measurable functions.

$(H_1)$The function $b(., x, \tilde{x}, y)$ is jointly measurable in $(x, \tilde{x}, y)$ and there exists a constant $C > 0$ such that

$$\|b(., x, \tilde{x}, y)\| \le C(1 + |x|) \quad for \quad any (x, \tilde{x}, y) \in \mathrm{R}^d \times \mathrm{R}^d \times \mathrm{R}^d$$

$(H_2)$The function $\sigma(., x, \tilde{x}, z)$ is jointly measurable in $(x, \tilde{x}, z)$ and there exists a constant $C > 0$ such that

$$\|\sigma(., x, \tilde{x}, z)\| \le C \qquad for \quad any (x, \tilde{x}, z) \in \mathrm{R}^d \times \mathrm{R}^d \times \mathrm{R}^d$$

$(H_3)$ The functions $b, \sigma$ satisfy a Lipschitz condition. That is, there exist constants $L_1, L_2$ such that for all $x_1, \tilde{x_1}, x_2, y_1, \tilde{y_1}, y_2$, and $t \in [0, \infty)$

$$\|b(t, x_1, \tilde{x_1}, y_1) - b(t, x_2, \tilde{x_2}, y_2)\|_{\mathrm{R}^d} \le L_1(|x_1 - x_2| + |\tilde{x_1} - \tilde{x_2}| + |y_1 - y_2|)$$

$$\|\sigma(t, x_1, \tilde{x_1}, y_1) - \sigma(t, x_2, \tilde{x_2} y_2)\|_{\mathrm{R}^d \times \mathrm{R}^l} \le L_2(|x_1 - x_2| + |\tilde{x_1} - \tilde{x_2}| + |y_1 - y_2|)$$

The existence of a unique solution of (3.1) which is $(\mathcal{F}_t)$-adapted and has $\alpha$-Hölder continuous sample paths, is ensured by $(H_0),(H_1),(H_2)$ , and standard results on existence and uniqueness of solutions of sdde with random coefficients by $(H_3)$. For $h\in\mathcal{H}$, $r\in L^{\frac{1}{1-\alpha}}([0,m],\mathrm{R}^d)$ and $u\in suppZ$ , let $F(h,r,u)(.)$ denote the unique solution to the dde

$$F(h)(t)=F(h)(0)+\int_0^t b(s,F(h)(s),F(h)(s-1),r(s))ds$$

$$+\int_0^t \sigma(s,F(h)(s),F(h)(s-1),u(s))\dot{h}(s)ds \quad t\in[0,\infty) \tag{3.2}$$

The existence of a unique solution of (3.2) which is $(\mathcal{F}_t)$-adapted is a consequence of the Lipschitz continuity of $\sigma$ and b and is standard.

Define $\tilde{I}:C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)\longrightarrow[0,\infty]$ by

$$\tilde{I}(\tilde{h})=\inf\{\frac{1}{2}\int_0^m|\dot{h}(s)|^2ds;h\in\mathcal{H}:$$

$$\exists(r,u)\in suppY\times suppZ \quad such \quad that \quad F(h,r,u)\equiv\tilde{h}\}$$

Otherwise, $\tilde{I}(\tilde{h})=+\infty$

Since $\tilde{I}$ is not necessarily lsc, we introduce its lsc regularization (see for example Bezuidenhout [2], p 651) $\tilde{I}^*$ defined by

$$\tilde{I}^*(\tilde{h})=\lim_{a\to 0}\inf_{\rho\in B_\alpha(\tilde{h},a)}\tilde{I}(\rho) \tag{3.3}$$

where $B_\alpha(\tilde{\rho},a)$ is the ball of radius $a$ centred at $\tilde{\rho}$ with respect to the norm $\|.\|_\alpha$. The existence of the limit on the right hand side of (3.3) is ensured by the fact that $\inf_{\rho\in B_\alpha(\tilde{h},a)}\tilde{I}(\rho)$ is a decreasing function of $a$.

The main result of the paper is the following.

***Theorem 3.1:*** *Let $\mu_\varepsilon$ be the law of $X^\varepsilon$ solution of (II) on $C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$, equipped with the norm $\|.\|_\alpha$. The family $\{\mu_\varepsilon,\varepsilon>0\}$ satisfies a large deviations principle with the following rate function*

$$\tilde{I}^*(f)=\lim_{a\to 0}\inf_{\rho\in B_\alpha(f,a)}\tilde{I}(\rho)$$

*where $\tilde{I}(\rho)=\inf\{\frac{1}{2}\int_0^m|\dot{h}(s)|^2ds;h\in\mathcal{H}:$*
*$\exists(r,u)\in suppY\times suppZ \quad such \quad that \quad F(h,r,u)\equiv\rho\}$*

*That is ,*

*(i) For any closed subset $C\subset C_\psi^{\alpha,0}([-1,m],\mathrm{R}^d)$*

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\mu_\varepsilon(C)\leq-\inf_{f\in C}\tilde{I}^*(f).$$

*(ii) For any open subset $G\subset C_\psi^{\alpha,0}([-1,m],\mathrm{R}^d)$*

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\mu_\varepsilon(C)\geq-\inf_{f\in G}\tilde{I}^*(f).$$

*where $\tilde{I}^*$ is defined in (3.3)and $\tilde{I}^*$ is a good rate function with respect to the topology of $C_\psi^{\alpha,0}([-1,m],\mathrm{R}^d),0<\alpha<\frac{1}{2}$.*

The proof is split into several lemmas. The rest of the paper is devoted to the proof of this result.

# 4. An extension of the contraction principle

Let $(E_X, d_X), (E_Y, d_Y), (E_Z, d_Z), (E', d')$ denote Polish spaces and $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that $\{X^\varepsilon, \varepsilon > 0\}$ is a family of random variables with value in $E_X$. Y is a random variable with values in $E_Y$. Z is a random variable with values in $E_Z$ .Given a rate function I on $E_X$ and $a > 0$ , set $\Gamma_a = \{x \in E_X : I(x) \le a\}$ and $\Gamma_\infty = \cup_a \Gamma_a$.

***Theorem 4.1 :*** *Let I be a good rate function on $E_X, F_N, F : \Gamma_\infty \times E_Y \times E_Z \longrightarrow E', X_N^\varepsilon, X^\varepsilon : \Omega \longrightarrow E'$, be applications such that the following hold :*
*(a)*
*(i)For all $a > 0$ and $N \ge 1$ , $F_N|_{\Gamma_a \times suppY \times suppZ}$ is continuous.*
*$F_N|_{\Gamma_a \times suppY \times suppZ}$ converges to $F|_{\Gamma_a \times suppY \times suppZ}$ uniformly as $N \longrightarrow \infty$.*
*(b) For each $a > 0$ and $N \ge 1$, $F_N(\{I \le a\} \times suppY \times suppZ)$ and*
*$F(\{I \le a\} \times suppY \times suppZ)$ are relatively compact in $(E', d')$.*
*(c) For all $N \ge 1, \{X_N^\varepsilon : \varepsilon > 0\}$ satisfies an LDP (as $\varepsilon \to 0$ ) on E' with good rate function*

$$I_N^*(\zeta) = \lim_{\rho \to 0} \inf_{\xi \in B'(\zeta, \rho)} I_N(\xi)$$

*where $B'(\zeta, \rho)$ denotes the ball of radius $\rho$ centred at $\zeta$ in $(E', d')$ , and*

$$I_N(\xi) = \inf\{I(x) : \exists (y, z) \in suppY \times suppZ \quad such \ \ that \ \ F_N(x, y, z) = \xi\}$$

***Proof:*** See Mellouk [10].

# 5. Proof of theorem 3.1

This section is devoted to proving main theorem by means of theorem 4.1. In order to apply theorem 4.1, we use the following notation :
For $0 < \alpha < \frac{1}{2}$, set
$(E_X, d_X) = (C_0^{\alpha,0}([0; m]; \mathrm{R}^l), \|.\|_\alpha)$,
$(E_Y, d_Y) = (L^{\frac{1}{(1-\alpha)}}([0, m], \mathrm{R}^d), \|.\|_{\frac{1}{L(1-\alpha)}})$,
$(E_Z, d_Z) = (C^{\alpha,0}([0; m]; \mathrm{R}^d), \|.\|_\alpha)$ ,
$(E', d') = (C_\psi^{\alpha,0}([-1; m]; \mathrm{R}^d), \|.\|_\alpha)$.
Throughout this section, $Y = \{Y(t), 0 \le t \le m\}$ and $Z = \{Z(t), 0 \le t \le m\}$ are the processes defined in section 3. The rate function I is defined by

$$I(h) = \{\frac{1}{2} \int_0^m |\dot{h}(s)|^2 ds, if h \in \mathcal{H}\}$$

Otherwise, $I(h) = +\infty$
In the following, for $a < \infty$, set $\Gamma_a = \{I \le a\}$ and $\Gamma_\infty = \cup_a \Gamma_a$. For $\varepsilon > 0$ and $N \ge 1$, set $\underline{t}_N = \frac{[Nt]}{N}$ ($[y]$ is the integer part of y).
Let $X^\varepsilon = \{X^\varepsilon(t), -1 < t < \infty\}$ be the solution of (II) and $X_N^\varepsilon = \{X_N^\varepsilon(t), -1 < t < \infty\}$ be the solution the sdde

$$dX_N^\varepsilon(t) = b(t, X_N^\varepsilon(t), X_N^\varepsilon(t-1), Y(t))dt$$
$$+ \varepsilon^{\frac{1}{2}} \sigma(\underline{t}_N, X_N^\varepsilon(\underline{t}_N), X_N^\varepsilon(\underline{t}_N - 1), Z(\underline{t}_N))dW_t \tag{5.1}$$

We have to consider two cases.

**case 1: b, $\sigma$ are bounded**

For $(r,u) \in L^{\frac{1}{(1-\alpha)}}([0,m],\mathrm{R}^d) \times C^{\alpha,0}([0,m];\mathrm{R}^d)$ , define the map
$F_N(.) : C_0^{\alpha,0}([0,m];\mathrm{R}^l) \times L^{\frac{1}{(1-\alpha)}}([0,m];\mathrm{R}^d) \times C^{\alpha,0}([0,m];\mathrm{R}^d) \longrightarrow C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$

$$\text{by}\begin{cases} F_N(\omega,r,u)(t) = \psi(t) & -1 \le t \le 0 \\ F_N(\omega,r,u)(t) = F_N(\omega,r,u)(\frac{k}{N}) & \\ \quad + \int_{\frac{k}{N}}^t b(s, F_N(\omega,r,u)(s), F_N(\omega,r,u)(s-1), r(s))ds & \\ \quad + \sigma(\frac{k}{N}, F_N(\omega,r,u)(\frac{k}{N}), F_N(\omega,r,u)(\frac{k}{N}-1), u(\frac{k}{N}))(\omega(t)-\omega(\frac{k}{N})) & \\ & for \quad \frac{k}{N} \le t \le \frac{k+1}{N} \end{cases}$$

Let $g \in \mathcal{H}$ and $e(g)$ defined as in (2.2). For g with $e(g) \le a$

$$\begin{cases} F_N(g,r,u)(t) = F_N(g,r,u)(\frac{k}{N}) & \\ + \int_0^t b(s, F_N(g,r,u)(s), F_N(g,r,u)(s-1), r(s))ds & \\ + \int_0^t \sigma(\frac{[Ns]}{N}), F_N(g,r,u)(\frac{[Ns]}{N}, F_N(g,r,u)(\frac{[Ns]}{N}-1), u(\frac{[Ns]}{N}))\dot{g}(s)ds & \\ & t \in [0,\infty) \\ F_N(g,r,u)(t) = \psi(t) & t \in [-1,0] \end{cases}$$

Notice that $X_N^\varepsilon(s) = F_N(\varepsilon^{\frac{1}{2}}W, Y, Z)(s)$ where W is standard brownian.
Define $I_N(f) = \inf\{\frac{1}{2}e(g); F_N(g) = f\}$ for each $f \in C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$.

**Case 2: for any b and $\sigma$**

We remove the boundedness assumption on b and $\sigma$.
For $R > 0$, define $m_R := \sup\{|b(t,x,\tilde{x},y)|, |\sigma(t,x,\tilde{x},y)|; t \in [0,m], |x| \le R, |\tilde{x}| \le R, |y| \le R\}$ and $b_R^i := (-m_R - 1) \vee b_i \wedge (m_R + 1)$,
$\sigma_{i,j}^R := (-m_R - 1) \vee \sigma_{i,j} \wedge (m_R + 1), 1 \le i,j \le d$.

Put $b_R := (b_1^R, b_2^R, ..., b_d^R)$ and $\sigma_R := (\sigma_{i,j}^R)_{1 \le i,j \le d}$.
Then $b_R(t,x,\tilde{x},y) = b(t,x,\tilde{x},y)$, $\sigma_R(t,x,\tilde{x},y) = \sigma(t,x,\tilde{x},y)$, for $t \in [0,m], |x| \le R, |\tilde{x}| \le R, |y| \le R$. Furthermore, $b_R$ and $\sigma_R$ satisfy the Lipschitz condition $(H_3)$ with the same Lipschitz constant.

If $g \in \mathcal{H}$ is absolutely continuous, set $e(g)$ as defined in (2.2).

For g with $e(g) < \infty$ , and for $(r,u) \in L^{\frac{1}{(1-\alpha)}}([0,m],\mathrm{R}^d) \times C^{\alpha,0}([0,m];\mathrm{R}^d)$ ,let $F_R(g,r,u)$ be the solution to the dde

$$\begin{cases} F_R(g,r,u)(t) = F_R(g,r,u)(0) & \\ \qquad + \int_0^t b(s, F_R(g,r,u)(s), F_R(g,r,u)(s-1), r(s))ds & \\ \qquad + \int_0^t \sigma(s, F_R(g,r,u)(s), F_R(g,r,u)(s-1), u(s))\dot{g}(s)ds & t \in (0,\infty) \\ X_t^\varepsilon = \psi(t) & t \in [-1,0] \end{cases}$$

Define $I_N(f) = \inf\{\frac{1}{2}e(g); F_R(g) = f\}$ for each $f \in C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$.

The existence and uniqueness of the solution of (5.1) follows from hypothesis $(H_0)$- $(H_3)$ on the coefficients. Furthermore, the trajectories of $X^\varepsilon$ and $X_N^\varepsilon$ belong almost surely to $C^{\alpha,0}([-1,m];\mathrm{R}^d)$.

To prove the central result : theorem 3.1 we will follow step by step the assumptions of theorem 4.1. To complete the argument,we will remind some results found in Mellouk [10](see also Y.J. Hu [9]). Precisely, these are lemma 5.1,5.2, 5.3, 5.4 and 5.5.

## 5.1 Continuity of $F_N$

We prove that
$F_N(.) : C_0^{\alpha,0}([0,m];\mathrm{R}^l) \times L^{\frac{1}{(1-\alpha)}}([0,m];\mathrm{R}^d) \times C^{\alpha,0}([0,m];\mathrm{R}^d) \longrightarrow C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$
(resp. $F_R$) is continuous . Fix $N \ge 1$ and let $(h_1,r_1,u_1),(h_2,r_2,u_2) \in C_0^{\alpha,0}([0,m];\mathrm{R}^l) \times$

$L^{\frac{1}{(1-\alpha)}}([0,m];\mathrm{R}^d)\times C^{\alpha,0}([0,m];\mathrm{R}^d)$ ; then set $F_N^{(i)}(.)=F_N(h_i,r_i,u_i), i=1,2$ and $\Psi_N(.)=F_N^{(1)}(.)-F_N^{(2)}(.)$.

***Lemma 5.1:*** *Given $C>0$, there exists a constant $C_N>0$(depending on N and C) such that , for $\|h_1\|_\alpha\vee\|h_2\|_\alpha\leq C$*

$$\|\Psi_N(.)\|_\alpha\leq C_N\|h_1-h_2\|_\alpha \tag{5.2}$$

## 5.2 Uniform convergence of $F_N$ to F on $\Gamma_a\times suppY\times suppZ$

To verify assertion (a)(ii) of theorem 4.1, we use the following:

***Lemma 5.2:*** *For any $a>0$,*

$$\sup_N\sup_{\|h\|_{\mathcal{H}}\leq a}(\|F_N(h)(.)\|_\infty\vee\|F(h)(.)\|_\infty)<\infty \tag{5.3}$$

$$\lim_{N\to\infty}\sup_{\|h\|_{\mathcal{H}}\leq a}(\|F_N(h)(.)-F(h)(.)\|_\alpha)=0 \tag{5.4}$$

## 5.3 Relative compactness

We prove the condition (b) of theorem 4.1 from the following:

***Lemma 5.3:*** *Let I be the good rate function defined by (2.3), $0<\alpha<\frac{1}{2}$ and K be a relatively compact subset of $C_0^{\alpha,0}([0,m];\mathrm{R}^l)$. Then for each $N\geq 1, a>0$, the sets $F_N(K\times suppY\times suppZ), F_N(\{I\leq a\}\times suppY\times suppZ)$ and $F(\{I\leq a\}\times suppY\times suppZ)$ are relatively compact in $C_\psi^{\alpha,0}([0,m];\mathrm{R}^d)$*

## 5.4 Large Deviation principle for $X_N^\varepsilon$ as $\varepsilon\to 0$

For $N\geq 1$, we prove that the family $X_N^\varepsilon\equiv F_N(\varepsilon^{\frac{1}{2}}W,Y,Z)$ defined by (5.1) satisfies on $C_0^{\alpha,0}([0,m];\mathrm{R}^d)$ an LDP, and show that the rate function is of the form (4.1). Since $F_N$ is continuous on $C_0^{\alpha,0}([0,m];\mathrm{R}^l)\times L^{\frac{1}{(1-\alpha)}}([0,m];\mathrm{R}^d)\times C^{\alpha,0}([0,m];\mathrm{R}^d)$ we use a version of the contraction principle. Schilder's theorem implies that $\varepsilon^{\frac{1}{2}}W$ satisfies an LDP on $C_0^{\alpha,0}([0,m];\mathrm{R}^l)$ with rate function I defined by (2.3).

For $N\geq 1$, define $I_N(f)=\inf\{I(h);h\in\mathcal{H}:$ $\exists(r,u)\in suppY\times suppZ$ *such that* $F_N(h,r,u)\equiv f\}$, and let $I_N^*$ be its lsc regularization, ie

$$I_N^*(f)=\lim_{a\to 0}\inf_{g\in B_\alpha(f,a)}I_N(g)$$

An argument similar to that in the proof of theorem 4.1 shows that $I_N^*$ is a good rate function and we check that $\{X_N^\varepsilon,\varepsilon>0\}$ satisfies an LDP with rate function $I_N^*$.

To prove lower and upper bound we use these following lemmas.

***Lemma 5.4:*** *Let G be an open subset of $C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$ ; then*

$$\liminf_{\varepsilon\to 0}\varepsilon\ln P(X_N^\varepsilon\in G)\geq-\inf\{I_N^*(f);f\in G\}$$

***Lemma 5.5:*** *Let A be a closed subset of $C_\psi^{\alpha,0}([-1,m];\mathrm{R}^d)$ ; then*

$$\limsup_{\varepsilon\to 0}\varepsilon\ln P(X_N^\varepsilon\in A)\leq-\inf\{I_N^*(f);f\in A\}$$

## 5.5 Exponential approximations

Finally, we show that $\{X^{\varepsilon}_{N}, \varepsilon > 0\}$ defined by (5.1) are exponentially good approximations of $\{X^{\varepsilon}, \varepsilon > 0\}$ defined by

$$X^{\varepsilon}(t) = X^{\varepsilon}(0) + \int_0^t b(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Y(s))ds$$

$$+ \varepsilon^{\frac{1}{2}} \int_0^t \sigma(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Z(s))dW_s$$

Let us at first establish the following approximation.

***Lemma 5.6:*** *For any* $\delta > 0$

$$\limsup_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(\sup_{t\in[0,m]} |X^{\varepsilon}_{N}(t) - X^{\varepsilon}(t)| > \frac{\delta}{N^{\alpha}}) = -\infty \tag{5.5}$$

***Proof:*** We prove (5.5) by induction on m. We first prove it for $m = 1$ . Since the drift coefficient b is not necessarily bounded, to prove (5.5) let us introduce some auxiliary results.

Let $0 < \alpha < \beta < \frac{1}{2}$ and $0 < \gamma < \beta - \alpha$; then by theorem 2.5

$$\limsup_{\varepsilon\to 0} \varepsilon \ln P(\sup_{1\leq k\leq N} |\varepsilon^{\frac{1}{2}}(W_{\frac{k}{N}} - W_{\frac{k-1}{N}})| \geq N^{\gamma-\beta})$$

$$\leq \limsup_{\varepsilon\to 0} \varepsilon \ln P(\|\varepsilon^{\frac{1}{2}} W\|_{\beta} \geq N^{\gamma})$$

$$\leq -\inf\{\frac{1}{2}\|h\|_{\mathcal{H}}; \|h\|_{\beta} \geq N^{\gamma}\} \leq -\frac{1}{2}N^{2\gamma} \tag{5.6}$$

Indeed, if $h \in \mathcal{H}$ satisfies $\|h\|_{\beta} \geq N^{\gamma}$ , the Cauchy-Schwarz inequality implies $\|h\|_{\mathcal{H}} \geq N^{\gamma}$. Define the set

$$B_{\beta,\gamma,\varepsilon} = \{\sup_{1\geq k\geq N} |\varepsilon^{\frac{1}{2}}(W_{\frac{k}{N}} - W_{\frac{k-1}{N}})| \leq N^{\gamma-\beta}\} \bigcap \{\|\varepsilon^{\frac{1}{2}} W\|_{\beta} \leq N^{\gamma}\} \tag{5.7}$$

by

$$\limsup_{\varepsilon\to 0} \varepsilon \ln P(\sup_{1\geq k\geq N} |\varepsilon^{\frac{1}{2}}(W_{\frac{k}{N}} - W_{\frac{k-1}{N}})| \leq N^{\gamma-\beta}) \leq -\frac{1}{2}N^{2\gamma}$$

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(B^{c}_{\beta,\gamma,\varepsilon}) = -\infty \tag{5.8}$$

Furthermore, on the set $B_{\beta,\gamma,\varepsilon}$ , by Gronwall's lemma and the assumptions of the coefficients $b, \sigma$ for $t \in [-1, m]$, we deduce the existence of a constant $C > 0$ such that

$$X^{\varepsilon}_{N} \leq C\{\sum_{k=1}^{N} \varepsilon^{\frac{1}{2}} |W_{\frac{k}{N\wedge t}} - W_{\frac{k-1}{N\wedge t}}| + \int_0^t \{1 + |X^{\varepsilon}_{N}(s)|\}ds\}$$

$$\leq C\{N^{\gamma+1-\beta} + \int_0^t |X^{\varepsilon}_{N}(s)|ds\}$$

As $N$ is large enough, there exists a constant $K>0$ such that $\int_0^t |X_N^\varepsilon(s)|ds \le K$ and $N^{\gamma+1-\beta} >> K$. So,there exists a constant $\check{C}>0$ such that

$$X_N^\varepsilon \le \check{C} N^{\gamma+1-\beta} \tag{5.9}$$

Set $\Psi_N^\varepsilon(.) = X_N^\varepsilon(.) - X^\varepsilon(.)$ and $\underline{t}_N = \frac{[Nt]}{N}$ then for $t\in[0,1]$ , $\Psi_N^\varepsilon(t)$ satisfies

$$\Psi_N^\varepsilon(t) = \int_0^t \{b(s, X_N^\varepsilon(s), X_N^\varepsilon(s-1), Y(s)) - b(s, X^\varepsilon(s), X^\varepsilon(s-1), Z(s))\}ds$$

$$+\varepsilon^{\frac{1}{2}} \int_0^t \{\sigma(\underline{s}_N, X_N^\varepsilon(\underline{s}_N), X_N^\varepsilon(s-1), Y(s)) - \sigma(s, X^\varepsilon(s), X^\varepsilon(s-1), Z(s))\}dW_s \tag{5.10}$$

For $\rho>0$ , we define $\tau_{N,\varepsilon}^\rho(\omega) = \inf\{t\ge 0; |X_N^\varepsilon(t,\omega) - X_N^\varepsilon(\underline{t}_N,\omega)| \ge \frac{\rho}{N^\alpha}\} \wedge 1$

$$\theta_{N,\varepsilon}^\rho(\omega) = \inf\{t\ge 0; |\Psi_N^\varepsilon(t,\omega)| \ge \frac{\delta}{N^\alpha}\} \wedge \tau_{N,\varepsilon}^\rho(\omega)$$

and $v_{N,\varepsilon}^\rho(t) = \int_\Omega \{\frac{\rho^2}{N^{2\alpha}} + |\Psi_N^\varepsilon(t\wedge\theta_{N,\varepsilon}^{\rho,\delta}(\omega),\omega)|^2\}^{\frac{1}{\varepsilon}} dP$
Then clearly,

$$P(\sup_{t\in[0,1]} |X_N^\varepsilon(t) - X^\varepsilon(t)| > \frac{\delta}{N^\alpha}) \le P(\tau_{N,\varepsilon}^\rho < 1) + P(\theta_{N,\varepsilon}^{\rho,\delta} < 1) \tag{5.11}$$

Observe that $P(\tau_{N,\varepsilon}^\rho<1) \le P(\tau_{N,\varepsilon}^\rho<1, B_{\beta,\gamma,\varepsilon}) + P(\tau_{N,\varepsilon}^\rho<1, B_{\beta,\gamma,\varepsilon}^c)$
First, we apply Stroock's inequality (2.4) and expression (5.9), together with hypothesis $(H_0)$- $(H_2)$ to obtain the existence of a constant $C>0$ such that

$$P(\tau_{N,\varepsilon}^\rho<1) \le \sum_{k=1}^N P(\sup_{\frac{k-1}{N}\le t\le\frac{k}{N}} |X_N^\varepsilon(t) - X_N^\varepsilon(\frac{k-1}{N})| \ge \frac{\rho}{N^\alpha}, B_{\beta,\gamma,\varepsilon}) + P(B_{\beta,\gamma,\varepsilon}^c)$$

$$\le CdN \exp\{-\frac{(\rho - CdN^{\gamma+1-\beta}(\frac{1}{N}^{1-\alpha}))^2}{C\varepsilon d^2(\frac{1}{N})^{1-2\alpha}}\} + P(B_{\beta,\gamma,\varepsilon}^c)$$

$$\le CdN \exp(-\frac{CN^{1-2\alpha}}{\varepsilon}) + P(B_{\beta,\gamma,\varepsilon}^c)$$

Since $N^{\gamma+\alpha-\beta} \longrightarrow 0$, as $\gamma<\beta-\alpha$, thus, using (5.11), we deduce

$$\lim_N \limsup_\varepsilon \varepsilon \ln P(\tau_{N,\varepsilon}^\rho<1) = -\infty \tag{5.12}$$

Since $suppZ$ is a compact subset of $C_0^{\alpha,0}([0,m];\mathrm{R}^d)$, for every $\rho>0$ there exists $N_0\ge 1$ such that, for $N\ge N_0$,

$$\sup_{0\le t\le 1} |Z(t) - Z(\underline{t}_N)| \le \rho N^{-\alpha} \tag{5.13}$$

For $0<\varepsilon<\frac{1}{2}$, set $\rho_N = \frac{\rho}{N^\alpha}$ and $f_{\varepsilon,\rho}(y) = (\rho_N^2 + |y|^2)^{\frac{1}{\varepsilon}}$
Then, an application of Ito's formula to $f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))$ yields that $f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t\wedge\theta_{N,\varepsilon}^{\rho,\delta})) - \int_0^{t\wedge\theta_{N,\varepsilon}^{\rho,\delta}} g_{\varepsilon,N}^\rho(s)ds - \rho_N^{\frac{2}{\varepsilon}}$ , is a martingale, where , if $<.,.>$ is the inner product in $\mathrm{R}^d$ ;

$$g_{\varepsilon,N}^\rho(t) = \frac{2}{\varepsilon}(\rho_N^2 + |\Psi_N^\varepsilon(t)|^2)^{\frac{1}{\varepsilon-1}}$$

$$\langle \Psi_N^\varepsilon(t), b(t, X_N^\varepsilon(t), X_N^\varepsilon(t-1), Y(t)) - b(t, X^\varepsilon(t), X^\varepsilon(t-1), Y(t))\rangle$$

$$+ \frac{2}{\varepsilon}(\frac{1}{\varepsilon} - 1)\varepsilon$$

$$\|(\sigma(t, X_N^\varepsilon(\underline{t}_N), X_N^\varepsilon(\underline{t}_N - 1), Z(\underline{t}_N)) - \sigma(t, X^\varepsilon(t), X^\varepsilon(t-1), Z(t)))^* \Psi_N^\varepsilon(t)\|^2$$
$$(\rho_N^2 + |\Psi_N^\varepsilon(t)|^2)^{\frac{1}{\varepsilon - 2}}$$

$$+ \|\sigma(\underline{t}_N, X_N^\varepsilon(\underline{t}_N), X_N^\varepsilon(\underline{t}_N - 1), Z(\underline{t}_N)) - \sigma(t, X^\varepsilon(t), X^\varepsilon(t-1), Z(t))\|^2$$
$$(\rho_N^2 + |\Psi_N^\varepsilon(t)|^2)^{\frac{1}{\varepsilon - 1}}$$

For $0 \le t \le \tau_{N,\varepsilon}^\rho$, using (5.13), we have, for $N \ge N_0$ and $0 < \varepsilon < \frac{1}{2}$ that there exists $C > 0$ such that

$$|g_{\varepsilon,N}^\rho(t)| \le C\frac{2}{\varepsilon}(\rho_N^2 + |\Psi_N^\varepsilon(t)|^2)^{\frac{1}{\varepsilon}} \frac{|\Psi_N^\varepsilon(t)|}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} |X_N^\varepsilon(t) - X^\varepsilon(t)|$$

$$+ C|\frac{1}{\varepsilon} - 1|\{\rho_N^2 + |Z(\underline{t}_N) - Z(t)|^2\} \frac{|\Psi_N^\varepsilon(t)|^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} (\rho_N^2 + |\Psi_N^\varepsilon(t)|^2)^{\frac{1}{\varepsilon - 1}}$$

$$+ C\{\rho_N^2 + |Z(\underline{t}_N) - Z(t)|^2\}(\rho_N^2 + |\Psi_N^\varepsilon(t)|^2)^{\frac{1}{\varepsilon - 1}}$$

$$\le C\frac{1}{\varepsilon} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t)) \frac{|\Psi_N^\varepsilon(t)|\rho_N}{\rho_N^2 + C|\Psi_N^\varepsilon(t)|^2}$$

$$+ |\frac{1}{\varepsilon} - 1| \frac{\rho_N^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} \frac{|\Psi_N^\varepsilon(t)|^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))$$

$$+ C|\frac{1}{\varepsilon} - 1| \frac{|Z(\underline{t}_N) - Z(t)|^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} \frac{|\Psi_N^\varepsilon(t)|^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))$$

$$+ C\frac{\rho_N^2 + |Z(\underline{t}_N) - Z(t)|^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))$$

$$\le C\{(\frac{1}{\varepsilon} + 1) + (1 + \frac{|Z(\underline{t}_N) - Z(t)|^2}{\rho_N^2 + |\Psi_N^\varepsilon(t)|^2})\} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))$$

$$\le C\frac{1}{\varepsilon} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))$$

This together with Doob's stopping theorem, shows that there exists a constant $\check{C} < \infty$ independent of $N, \varepsilon, \rho$ and $N_0$ such that, for $N \geq N_0$

$$\upsilon^{\rho}_{N,\varepsilon}(t) \leq \rho_N^{\frac{2}{\varepsilon}} + \frac{\check{C}}{\varepsilon}\int_0^t \upsilon^{\rho}_{N,\varepsilon}(s)ds \qquad t \in [0,1]$$

(see, for example, Deuschel and Stroock [6], p. 30)

Therefore, for $N \geq N_0$, $\upsilon^{\rho}_{N,\varepsilon}(1) \leq \exp\{\frac{1}{2}(\check{C} + 2\ln\rho - 2\alpha\ln N)\}$

Since for all $N \geq 1$, $P(\theta^{\rho,\delta}_{N,\varepsilon} < 1) \leq (\frac{\rho^2+\delta^2}{N^{2\alpha}})^{-\frac{1}{\varepsilon}}\upsilon^{\rho}_{N,\varepsilon}(1)$

We conclude

$$\lim_{\rho\to 0}\sup_N \limsup_{\varepsilon\to 0} \varepsilon \ln P(\theta^{\rho,\delta}_{N,\varepsilon} < 1) = -\infty$$

This together with (5.11) and (5.12) implies (5.5) for $m = 1$.

Assume now (5.5) holds for some integer m. We will prove it is also true for $m + 1$.

Let $\Psi^{\varepsilon}_N, \tau^{\varepsilon}_{N,\rho}$ be defined as before.

**Case 1($b$ *and* $\sigma$ are bounded)**

In addition, we introduce two new stopping times.

$$\tau^{1,\varepsilon}_{N,\rho} := \inf\{t \geq 0; |X^{\varepsilon}(t-1) - X^{\varepsilon}_N(t-1)| \geq \frac{\rho}{N^{\alpha}}\}$$

$$\tau^{2,\varepsilon}_{N,\rho} := \inf\{t \geq 0; |X^{\varepsilon}_N(t-1) - X^{\varepsilon}_N(\frac{t_N}{N} - 1)| \geq \frac{\rho}{N^{\alpha}}\}$$

and define $\Phi^{\varepsilon}_{N,\rho}(t) = \Psi^{\varepsilon}_N(t \wedge \tau^{1,\varepsilon}_{N,\rho} \wedge \tau^{2,\varepsilon}_{N,\rho} \wedge \tau^{\varepsilon}_{N,\rho})$ and
$\bar{\theta}^{\varepsilon}_{N,\rho} := \inf\{t \geq 0, |\Phi^{\varepsilon}_{N,\rho}(t)| \geq \frac{\delta}{N^{\alpha}}\}$

Then,

$$P(\sup_{t\leq m+1} |\Psi^{\varepsilon}_N(t)| > \frac{\delta}{N^{\alpha}})$$

$$\leq P(\tau^{1,\varepsilon}_{N,\rho} \wedge \tau^{2,\varepsilon}_{N,\rho} \wedge \tau^{\varepsilon}_{N,\rho} \geq m+1)$$

$$+ P(\sup_{t\leq m+1} |X^{\varepsilon}(t) - X^{\varepsilon}_N(t)| > \frac{\delta}{N^{\alpha}}, \tau^{1,\varepsilon}_{N,\rho} \wedge \tau^{2,\varepsilon}_{N,\rho} \wedge \tau^{\varepsilon}_{N,\rho} > m+1)$$

$$\leq P(\tau^{1,\varepsilon}_{N,\rho} \leq m+1) + P(\tau^{2,\varepsilon}_{N,\rho} \wedge \tau^{\varepsilon}_{N,\rho} \leq m+1) + P(\bar{\theta}^{\varepsilon}_{N,\rho} \leq m+1) \tag{5.14}$$

Then,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(\tau^{\varepsilon}_{N,\rho} \wedge \tau^{2,\varepsilon}_{N,\rho} \leq m+1) = -\infty \tag{5.15}$$

By the induction hypothesis,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(\tau^{1,\varepsilon}_{N,\rho} \leq m+1)$$

$$\leq \lim_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(\sup_{-1\leq t\leq m} |X^{\varepsilon}(t) - X^{\varepsilon}_N(t)| > \frac{\rho}{N^{\alpha}}) = -\infty \tag{5.16}$$

Using (5.14) and (5.15) and following the argument for $m = 1$, we see that (5.5) is also true for $m + 1$. □

**Case 2($b, \sigma$ are not bounded)**

Set $\Psi^{\varepsilon}_R(t) = X^{\varepsilon}(t) - X^{\varepsilon}_R(t)$ For $R_1 > 0$, define $\theta^{\varepsilon}_{R_1} := \inf\{t \geq 0; |X^{\varepsilon}(t)| \geq \frac{R_1}{N^{\alpha}}\}$

For any $R \geq R_1$, we have

$$\Psi^{\varepsilon}_R(t\wedge\theta^{\varepsilon}_{R_1}) = \int_0^{t\wedge\theta^{\varepsilon}_{R_1}} [b_R(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Y(s)) - b_R(s, X^{\varepsilon}_R(s), X^{\varepsilon}_R(s-1), Y(s))]ds$$

$$+\varepsilon^{\frac{1}{2}} \int_0^{t\wedge\theta^{\varepsilon}_{R_1}} [\sigma_R(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Z(s)) - \sigma_R(s, X^{\varepsilon}_R(s), X^{\varepsilon}_R(s-1), Z(s))]dW_s \quad (5.17)$$

For $\rho > 0$, let $\upsilon_{\lambda}(y) = (\rho^2 + |y|^2)^{\lambda}$ and

$$\tau^{\varepsilon}_{R,\rho} := \inf\{t \geq 0, |X^{\varepsilon}(t-1) - X^{\varepsilon}_R(t-1)| \geq \frac{\rho}{N^{\alpha}}\}$$

$$\theta^{\varepsilon}_{R,\rho} := \inf\{t \geq 0, |\Psi^{\varepsilon}_{R,\rho}(t)| \geq \frac{\delta}{N^{\alpha}}\}$$

where $\Psi^{\varepsilon}_{R,\rho}(t) := \Psi^{\varepsilon}_R(t \wedge \theta^{\varepsilon}_{R_1} \wedge \tau^{\varepsilon}_{R,\rho})$

Then,

$$P(\sup_{-1\leq t\leq m+1} |\Psi^{\varepsilon}_R(t)| > \frac{R_1}{N^{\alpha}}) \leq P(\theta^{\varepsilon}_{R_1} \leq m+1) + P(\theta^{\varepsilon}_{R,\rho} \leq m+1) + P(\tau^{\varepsilon}_{R,\rho} \leq m+1)$$

$$\leq P(\sup_{-1\leq t\leq m+1} |X^{\varepsilon}(t)| > \frac{R_1}{N^{\alpha}})$$

$$+ P(\sup_{-1\leq t\leq m+1} |X^{\varepsilon}(t) - X^{\varepsilon}_R(t)| > \frac{\rho}{N^{\alpha}}) + P(\theta^{\varepsilon}_{R,\rho} \leq m+1) \quad (5.18)$$

By the induction hypothesis

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(\sup_{-1\leq t\leq m} |X^{\varepsilon}(t) - X^{\varepsilon}_R(t)| > \frac{\rho}{N^{\alpha}}) = -\infty \quad (5.19)$$

By Ito's formula

$$\Phi_{\lambda}(\Psi^{\varepsilon}_{R,\rho}(t)) - \int_0^{t\wedge\theta^{\varepsilon}_{R_1}\wedge\tau^{\varepsilon}_{R,\rho}} \gamma^{\varepsilon}_{\lambda}(s)ds - \rho^{2\lambda} \quad (5.20)$$

is a martingale with initial value zero where

$$\gamma^{\varepsilon}_{\lambda}(s) = 2\lambda(\rho^2 + |\Psi^{\varepsilon}_R(s)|^2)^{\lambda-1}$$

$$\langle\Psi^{\varepsilon}_R(s), b(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Y(s)) - b(s, X^{\varepsilon}_R(s), X^{\varepsilon}_R(s-1), Y(s))\rangle$$

$$+ 2\lambda(\lambda-1)\varepsilon(\rho^2 + |\Psi^{\varepsilon}_R(s)|^2)^{\lambda-2}$$

$$|(\sigma(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Z(s)) - \sigma(\underline{s}_N, X^{\varepsilon}_R(\underline{s}_N), X^{\varepsilon}_R(\underline{s}_N-1), Z(\underline{s}_N)))^*\Psi^{\varepsilon}_R(s)|^2$$

$$+ \lambda\varepsilon(\rho^2 + |\Psi^{\varepsilon}_R(s)|^2)^{\lambda-2}$$

$$\|\sigma(s, X^{\varepsilon}(s), X^{\varepsilon}(s-1), Z(s)) - \sigma(\underline{s}_N, X^{\varepsilon}_R(\underline{s}_N), X^{\varepsilon}_R(\underline{s}_N-1), Z(\underline{s}_N))\|^2_{H.S}$$

For $s \leq t \wedge \theta^{\varepsilon}_{R_1} \wedge \tau^{\varepsilon}_{R,\rho}$

$$\gamma^{\varepsilon}_{\lambda}(s) \leq C(\lambda + \lambda(\lambda+1)\varepsilon)(\rho^2 + |\Psi^{\varepsilon}_{R}(s)|^2)^{\lambda}$$

Choose $\lambda = \frac{1}{2}$, and take expectations in (5.20) to obtain

$$E[(\rho^2 + |\Psi^{\varepsilon}_{R,\rho}(t)|^2)^{\frac{1}{\varepsilon}}] \leq \rho^{\frac{2}{\varepsilon}} + \frac{C}{\varepsilon}\int_0^t E[(\rho^2 + |\Psi^{\varepsilon}_{R,\rho}(s)|^2)^{\frac{1}{\varepsilon}}]ds$$

Hence,

$$E[(\rho^2 + |\Psi^{\varepsilon}_{R,\rho}(t)|^2)^{\frac{1}{\varepsilon}}] \leq \rho^{\frac{2}{\varepsilon}} \exp^{\frac{Ct}{\varepsilon}}$$

Since,

$$(\rho^2 + \delta^2)^{\frac{1}{\varepsilon}} P(\theta^{\varepsilon}_{R,\rho} \leq m+1) \leq E[(\rho^2 + |\Psi^{\varepsilon}_{R,\rho}(m+1)|^2)^{\frac{1}{\varepsilon}}]$$

we have

$$P(\theta^{\varepsilon}_{R,\rho} \leq m+1) \leq (\frac{\rho^2}{\rho^2 + \delta^2})^{\frac{1}{\varepsilon}} \exp^{\frac{C}{\varepsilon}(m+1)}$$

Therefore

$$\limsup_{\varepsilon \to 0} \varepsilon \ln P(\theta^{\varepsilon}_{R,\rho} \leq 1) \leq \ln(\frac{\rho^2}{\rho^2 + \delta^2}) + C$$

Given $M > 0$, choose $\rho$ small enough so that $\ln(\frac{\rho^2}{\rho^2+\delta^2}) + C \leq -2M$
Choose $R_0$ so that

$$\limsup_{\varepsilon \to 0} \varepsilon \ln P(\tau^{\varepsilon}_{R,\rho} \leq 1) \leq -2M \quad for R \geq R_0$$

So, we have

$$\lim_{R\to\infty} \limsup_{\varepsilon \to 0} \varepsilon \ln P(\sup_{-1 \leq t \leq m+1} |X^{\varepsilon}(t) - X^{\varepsilon}_{R}(t)| > \frac{\delta}{R^{\alpha}}) \leq -M$$

Since M is arbitrary, it follows from (5.17), (5.18), (5.19) that

$$\lim_{R\to\infty} \limsup_{\varepsilon \to 0} \varepsilon \ln P(\sup_{-1 \leq t \leq m+1} |\Psi^{\varepsilon}_{R}(t)| > \frac{\delta}{R^{\alpha}})$$

$$\leq \limsup_{\varepsilon \to 0} \varepsilon \ln P(\sup_{-1 \leq t \leq m+1} |X^{\varepsilon}(t)| > R_1) \vee \{\ln(\frac{\rho^2}{\rho^2 + \delta^2}) + C\}$$

By using Proposition (3.5) in [12], letting first $\rho \to 0$ and then $R_1$, we obtain (5.5) for $m+1$. □

***Lemma 5.8:*** *For any* $\delta > 0$

$$\limsup_{N\to\infty} \limsup_{\varepsilon \to 0} \varepsilon \ln P(\|X^{\varepsilon}_{N}(t) - X^{\varepsilon}(t)\|_{\alpha} \geq \delta) = -\infty$$

***Proof:*** See Mellouk [10].

**Remark:**

1- If b and $\sigma$ are independent to the random variables Y and Z, we have a generalization of LDP for stochastic systems with memory in Hölder topology (refer [10]).

2- If b and $\sigma$ are dependent to Y and Z, and the equation has no delay, we have

a LDP for random evolution equations ([8], [13]).
3- LDP found here can be extended to the case where there are several delays. As example, two delays $\tau_1$, $\tau_2$ are allowed in the equation (II), and we have:

$$\text{(III)}\begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon, X_{t-\tau_1}^\varepsilon, Y_t, Y_{t-\tau_1})dt + \varepsilon^{\frac{1}{2}}\sigma(t, X_t^\varepsilon, X_{t-\tau_2}^\varepsilon, Z_t, Z_{t-\tau_2})dW_t \\ \qquad\qquad t \in (0, \infty) \\ X_t^\varepsilon = \psi(t) \qquad\qquad t \in [-\tau, 0] \end{cases}$$

Here, we consider $\tau = \tau_1 \vee \tau_2$ then we use the same argument.
It would be interesting to extend this result to another strong topology such as Besov space.

## References

[1] P. Baldi, G. Ben Arous, and G. Kerkyacharian, Large deviation and Strassen law in Hölder norm. *Stochastic Process. Appl.*, **42**,(1992), 171-180.
[2] C. Bezuidenhout, A Large Deviations Principle for small perturbation of random evolution equations. *Ann. Probab.*, **15**,(1987), 646-658.
[3] H. Chernoff, A measure of asymptotic efficiency for tests of hypothesis based on the sum of observations. *Ann. Math. Stat.*, **23**, (1952),493-507.
[4] H. Cramer, Sur un nouveau théorème-limite de la théorie des probabilités, *Actualités scientifiques et industrielles*, **736**,(1938), 5-23.
[5] A. Dembo, and O. Zeitouni, *Large Deviations Techniques and Applications.* Boston: Jones and Bartlett (1993).
[6] J. D. Deuschel, and D.W. Stroock, Large Deviations. *Academic Press*, Boston, San Diego , New York,(1989).
[7] Elsanosi and al, Some solvable stochastic control problems with delay, *Stochastics*, **71**, (2000),69-89.
[8] M.I. Freidlin, and A.D. Wentzell, *Random perturbations of dynamical systems*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, (2000).
[9] Y. J. Hu, A large deviations principle for small random perturbation of random evolutions in Hölder norm. *Stoch. Process Appl.*, **68**, (1997), 83-99.
[10] M. Mellouk, A large Deviations Principle for random evolution equations. *Bernoulli*, **6**(6),(2000), 977-999.
[11] S-E A. Mohammed, and T. Zhang, Large Deviations for stochastic systems with memory. *Discrete and Continuous dynamical systems- serie B*, **6**(4),(2006), 881-893.
[12] S-E A. Mohammed, Stochastic differential systems with memory :theory, examples and applications, *Stochastic analysis and related topics VI*, Boston, (1998), 1-77.
[13] L. I. Rajaonarison, and T. J. Rabeherimanana, Nouveaux résultats sur les petites perturbations d'équations d'évolutions aléatoires, *Ann. Math. Blaise Pascal***19**, (2012), 276-296.
[14] D. Ruelle, *Statistical Mechanics, Rigorous Results.* New York : Benjamin, (1969).
[15] I. N. Sanov, On the probability of large deviations of random magnitudes, *Mat. Sb*, N.S. **42**,(1957), 11-44.
[16] M. Scheutzow, Qualitative behaviour of stochastic delay equations with a bounded memory. *Stochastics*, **12**, (1984), 41-80.
[17] M. Schilder, Some asymptotic formulas for Wiener integral, *Trans. Amer.*

*Math. Soc.*, **125**,(1966), 63-85.
[18] S. R. S. Varadhan, Diffusion processes in a small time interval, *Comm. Pure Appl. Math*, **20**,(1967) , 659-685.