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On Finslerian connections

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Abstract: Given a connection without torsion, we propose a method to know if the connection is Finslerian, in the case of a linear connection if it is Riemannian.

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1 Introduction

Let M be a paracompact differentiable manifold of dimension $n \geq 2$ and of class $\mathcal{C}^\infty$. The considered connection is an almost product structure [4] which is a version of that of [6] using the formalism of [3].
Let J be the natural tangent structure of tangent bundle TM, S a spray of class $\mathcal{C}^\infty$ on $TM - \{0\}$, $\mathcal{C}^1$ on null section, homogeneous of degree 1 ($[C, S] = S$), C is the Liouville field on the tangent bundle TM. The almost product structure $\Gamma = [J, S]$ is a connection without torsion. If S is of class $\mathcal{C}^2$ on the null section, then the connection Γ is a linear connection without torsion. The curvature of Γ is the Nijenhuis tensor of h, $R = \frac{1}{2}[h, h]$, with $h = \frac{I+\Gamma}{2}$, I being the identity vector 1−form.

2 Finslerian manifold

A map E from $TM - \{0\}$ in $\mathbb{R}^+$ with $E(0) = 0$, of class $\mathcal{C}^\infty$ on $\mathcal{T}M$, of class $\mathcal{C}^1$ on the null section and homogeneous of degree 2 such that $dd_J E$, where d is the exterior derivation, has a maximal rank, defines a Finslerian structure. The map E is called an energy function. If E is of class $\mathcal{C}^2$ on null section, the manifold is Riemannian. The almost product structure, called the canonical connection [4] is given by

$$\Gamma = [J, S],$$

where S is a spray defined by [5]

$$i_S dd_J E = -dE,$$

i_S being the inner product with respect to S. This connection is without torsion and conservative [4].
The fundamental form $\Omega = dd_J E$ allows to define, a metric g on the vertical bundle by

$$g(JX, JY) = \Omega(JX, Y),$$

for all X, $Y \in \chi(TM)$, where $\chi(TM)$ denotes the set of all vector fields on TM.
There is [4], one and only one metric lift D of a canonical connection such that:

1. $J\mathbb{T}(hX, hY) = 0$,
2. $\mathbb{T}(JX, JY) = 0$ ($\mathbb{T}(X, Y) = D_X Y - D_Y X - [X, Y]$),
3. $DJ = 0$,
4. $DC = v$,
5. $D\Gamma = 0$,
6. $Dg = 0$.

The linear connection D is called Cartan connection.

Proposition 2.1 ([1]). *Let E be an energy function, Γ a connection such that $\Gamma = [J, S]$. The following two relationships are equivalent: i) $-i_S dd_J E = dE$, ii) $d_h E = 0$.*

Remark 2.2 ([4]). This property expresses the fact that the energy function is conserved by parallel transport and the paths of the spray are the solutions of the "Lagrange equations" of energy E.

Proposition 2.3. *For the connection Γ satisfying the proposition 2.1, the scalar 1–form $d_v E$ is completely integrable.*

Proof. The Kernel of $d_v E$ is formed by vector fields belonging to the horizontal space Imh ($v \circ h = 0$) and vertical vector fields JY such that $L_{JY} E = 0$, $Y \in Imh$, taking into account $vJ = J$, L_{JY} being the Lie derivative with respect to a vector field JY.
As we have

$$[hX, hY] = h[hX, hY] + v[hX, hY] = h[hX, hY] + R(X, Y),$$

for all X, $Y \in \chi(TM)$, and that $d_h E = 0$ implies $d_R E = 0$. We obtain

$$[hX, hY] \in Ker d_v E.$$

It remains to show that $L_{v[hX, JY]} E = 0$, $\forall X \in Imh$, $Y \in Imh$ satisfying $L_{JY} E = 0$. This is immediate since we have $v = I - h$. □

Theorem 2.4. *Let $\Gamma = [J, S]$ be a connection. The connection Γ comes from a energy function if and only if*

1) there is an energy function E_0 such that $d_R E_0 = 0$.

2) the scalar 1–form $d_v E_0$ is completely integrable.

Then, there exists a constant $\varphi(x)$ on the bundle such that $e^{\varphi(x)} E_0$ is the energy function of Γ.

Proof. This is a consequence of the propositions 2.1,2.3 and the proof of theorem 1 of [1]. □

3 Riemannian manifold

In this paragraph, we assure that the energy function E is of class $\mathcal{C}^2$ on the null section, then the manifold (M,E) becomes Riemannian. With the Cartan connection D, we have [4]

$$D_{JX}JY = [J,JY]X, \ D_{hX}JY = [h,JY]X.$$

With the linear connection D, we associate a curvature

$$\mathcal{R}(X,Y)Z = D_{hX}D_{hY}JZ - D_{hY}D_{hX}JZ - D_{[hX,hY]}JZ \tag{3.1}$$

for all X, Y, $Z \in \chi(TM)$. The relationship between the curvature $\mathcal{R}$ and R is

$$\mathcal{R}(X,Y)Z = J[Z,R(X,Y)] - [JZ,R(X,Y)] + R([JZ,X],Y) + R(X,[JZ,Y]).$$

for all X, Y, $Z \in \chi(TM)$.
In particular,

$$\mathcal{R}(X,Y)S = -R(X,Y).$$

As the function E is homogeneous of degree 2 and of class $\mathcal{C}^2$ null on null section. On an open set U of M, $(x^i, y^j) \in TU$, the energy function is written

$$E = \frac{1}{2} g_{ij}(x^1, \ldots, x^n) y^i y^j,$$

where $g_{ij}(x^1, \ldots, x^n)$ are symmetric positive functions such that the matrix $(g_{ij}(x^1, \ldots, x^n))$ is invertible. And the relation $i_S dd_J E = -dE$ gives the spray S

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x^1, \ldots, x^n, y^1, \ldots, y^n) \frac{\partial}{\partial y^i},$$

we denote

$$\gamma_{ikj} = \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

and

$$\gamma_{ij}^k = g^{kl} \gamma_{ilj},$$

we have

$$G^k = \frac{1}{2} y^i y^j \gamma_{ij}^k.$$

We note $\Gamma_i^j(x,y) = y^l \Gamma_{il}^j(x)$, the horizontal projector is written

$$\begin{cases} h(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j} \\ h(\frac{\partial}{\partial y^j}) = 0 \end{cases}.$$

The vertical projector becomes

$$\begin{cases} v(\frac{\partial}{\partial x^i}) = \Gamma_i^j \frac{\partial}{\partial y^j} \\ v(\frac{\partial}{\partial y^j}) = \frac{\partial}{\partial y^j} \end{cases}.$$

The curvature $R = \frac{1}{2}[h,h]$ is then

$$R = \frac{1}{2} R_{ij}^k dx^i \wedge dx^j \otimes \frac{\partial}{\partial y^k} \text{ with } R_{ij}^k = \frac{\partial \Gamma_i^k}{\partial x^j} - \frac{\partial \Gamma_j^k}{\partial x^i} + \Gamma_i^l \frac{\partial \Gamma_j^k}{\partial y^l} - \Gamma_j^l \frac{\partial \Gamma_i^k}{\partial y^l},\ i,j,k,l \in \{1,\ldots,n\}.$$

Proposition 3.1 ([2])**.** *On a Riemannian manifold (M,E), the horizontal nullity space of the curvature R is generated as a module by the projectable vector fields belonging to this nullity space and, orthogonal to the image space ImR of the curvature R and $hN_R = hN_{\mathcal{R}}$.*

Proposition 3.2. *On a Riemannian manifold (M,E), the horizontal space and the space ImR of the curvature R generate a Lie algebra on $\mathcal{F}(TM)$ whose the vertical space is orthogonal to $JN_R + \langle C \rangle_{\mathcal{F}(TM)}$.*

Proof. It is immediate to see that ImR is orthogonal to $JN_R + \langle C \rangle_{\mathcal{F}(TM)}$, from the relations between the curvature R, $\mathcal{R}$ and hN_R [1].
Let be $X,\ Y \in \chi(TM)$, we have

$$[hX,hY] = h[hX,hY] + v[hX,hY] = h[hX,hY] + R(X,Y),$$

ie, $[hX,hY] \in Imh + ImR$, for all $X,\ Y \in \chi(TM)$.
Let be $X \in hN_R$ and $JY \perp JN_R + \{C\}$. We have $g(JY,JX) = 0$. We can consider X projectable according to the proposition 3.1 since g is bilinear. Given $Dg = 0$, we can write

$$D_{hZ} g(JY,JX) - g(D_{hZ}JY,JX) - g(JY,D_{JZ}JX) = 0.$$

From the relation $h^2 = h$, we get

$$D_{hZ}JY = v[hZ,JY] \text{ and } D_{hZ}JX = v[hZ,JX] = [h,JX]hZ = 0,$$

according to the proposition 4 [1]. The relation $v[hZ,JY]$ is orthogonal to JN_R. Similarly, $v[hZ,JY]$ is orthogonal to $C = JS$, given $[C,h] = 0$. If $JY,\ JZ$ are orthogonal to JN_R, we have $g(JY,JX) = 0$, $g(JZ,JX) = 0$ for all $X \in hN_R$.
From the relation $Dg = 0$, we can write $g(D_{hZ}JY,JX) + g(JY,D_{JZ}JX) = 0$ and $D_{JZ}JY = [J,JY]Z = J[JZ,Y]$, taking into account $[J,J] = 0$. Likewise, we find $D_{JZ}JX = J[JZ,X]$. We then have $g([JZ,JY],JX) = g(J[JZ,Y] + J[Z,JY],JX) = 0$. That is, $[JZ,JY]$ is orthogonal to JN_R. If $JY,\ JZ$ are orthogonal to C, it means $L_{JY}E = 0$ and $L_{JZ}E = 0$. We then have $L_{[JY,JZ]}E = 0$. Hence the result. □

Remark 3.3. For a Riemannian manifold, condition 2) of the theorem 2.4 could become $H + ImR$ completely integrable by the proposition 3.2 if $hN_R = \{0\}$.
An energy function is written

$$E_0 = \frac{1}{2} g_{ij}^0 y^i y^j.$$

Thus, the relation $d_R E = 0$ is equivalent to the following system of equations

$$\begin{cases} g_{kl}^0 R_{l,ij}^k = 0 \\ g_{kl}^0 R_{r,ij}^k = -g_{kr}^0 R_{l,ij}^k \quad with\ l \neq r. \end{cases}$$

In the matrix form, the system is written

$$\begin{pmatrix} g_{11}^0 & \cdots & g_{n1}^0 \\ \vdots & \ddots & \vdots \\ g_{1n}^0 & \cdots & g_{nn}^0 \end{pmatrix} \begin{pmatrix} R_{1,ij}^1 & \cdots & R_{n,ij}^1 \\ \vdots & \ddots & \vdots \\ R_{1,ij}^n & \cdots & R_{n,ij}^n \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & \ddots & -{}^t\mathbf{A} & \\ & \mathbf{A} & \ddots & \\ & & & 0 \end{pmatrix}.$$

Remark 3.4. In [7], an answer was given. Let M be a paracompact differentiable manifold, and D a linear connection on M, without torsion. For D to come from a Riemannian structure, it is necessary and sufficient that its groups of holonomy are relatively compact.

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