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SOME RESULTS ON LIE ALGEBRAS FROM CONNECTIONS

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ABSTRACT. In this paper, we consider a smooth manifold M of dimension n, $\mathcal{T}M$ denotes the tangent bundle of M without the null section. Our main goal is to see some remarkable algebraic properties of the Lie algebra $\mathfrak{A}_\Gamma$ of vector fields on $\mathcal{T}M$ whose the Lie derivatives vanish a Grifone's connection Γ, those of the curvature nullity space of Γ and those of the set of all Grifone's connections on M, by global ways such as the study of ($m \geq 2$)-derivations, the Chevalley-Eilenberg cohomology, of the Lie algebras generated by these considerations, and by local methods such that relations becoming from system of partial differential equations between all elements of these Lie algebras which permit their local forms to be well known. Especially we compute all ($m \geq 2$)-derivations, cohomology of the Lie algebra of infinitesimal automorphisms of Γ and of its normalizer both sub-algebras of $\mathfrak{A}_\Gamma$.

1. Introduction

Let M be a smooth manifold of dimension n and we denote TM its tangent bundle where $\mathcal{T}M$ the bundle TM without null section. On M we define a connection of Grifone Γ, that is to say a smooth 1-vector form on $\mathcal{T}M$ such that $J\Gamma = J$ and $\Gamma J = -J$ with J the tangent structure on TM cf. [11]. Give this connection is equivalent to the existence of direct split of the bundle TTM of TM to $H(TM) \oplus V(TM)$ where $h = \frac{I+\Gamma}{2}$ the horizontal projection, $v = \frac{I-\Gamma}{2}$ the vertical projection of Γ and $H(TM) = Im(h)$, $V(TM) = Im(v)$. The connection Γ defines an almost product structure such that $\Gamma^2 = I$ where I is the identity mapping. It permits h to be associated to the eigenvalue 1 and v to the another -1 of Γ. We note by $\mathfrak{A}_\Gamma$ the Lie algebra of all vector fields in $\mathcal{T}M$ vanishing Γ by Lie derivative, $\mathfrak{N}_R$ the nullity space of the curvature $R = -\frac{1}{2}[h, h]$ of Γ. That is $\mathfrak{N}_R$ is the set of vector fields on $\mathcal{T}M$ which vanish the 2-vector form R in the first argument for all vector fields on $\mathcal{T}M$ in the second one. To make this paper easy to read, we divide it into three different sections.

The first section consists in the study of the Lie algebras relative to $\mathfrak{A}_\Gamma$ and to $\mathfrak{N}_R$ by relatively global methods. In [11], we have studied derivations of some Lie algebras associated to $\mathfrak{A}_\Gamma$ and to $\mathfrak{N}_R$, such as the horizontal parts $\mathfrak{A}^h_\Gamma$ of $\mathfrak{A}_\Gamma$ and $\mathfrak{N}^h_R$ of $\mathfrak{N}_R$, and computed the first spaces of cohomology of Chevalley-Eilenberg which come from. One generalization of derivations of a Lie algebra is ($m \geq 2$)-derivations where $m \geq 2$ an integer, having the following properties: the Lie algebra of all derivations is in the one of all ($m > 2$)-derivations (m fixed), we obtain the triviality for $m = 2$. But the converse is

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false in general cf. one result of [10]. We did calculus of all $(m > 2)$-derivations of these Lie algebras in [13] and they were derivations. We don't have computed all derivations of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R = \mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$ because of those of vertical part $\mathfrak{A}_\Gamma^v$ of $\mathfrak{A}_\Gamma$ in [11] don't have known yet. Here we find the Lie algebra of derivations $Der\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right)$ of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ and in addition $Der^m\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right)$ that of all $(m > 2)$-derivations of this algebra. We prove that for all $m > 2$, $Der^m\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right) = Der\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right) + \mathcal{L}\left(A_1, A_2\right)$ where $\mathfrak{A}_\Gamma^v = A_1 + A_2$ is a semi-direct sum of Lie algebras by theorem of Hermann on the integrability of $\mathfrak{A}_{H^0}$ the Lie algebra spanned by all projectable horizontal vector fields on $\mathcal{T}M$ and by a proposition of [11] stating that $\mathfrak{A}_\Gamma^v$ is the centralizer of $\mathfrak{A}_{H^0}$ ($\mathcal{L}\left(A_1, A_2\right)$ the set of linear mappings from A_1 to A_2). Here, we say one important consequence of this theorem which connects the m-derivations of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ with geometric property of the connection Γ, that is one necessary and sufficient condition for Γ to be flat is $Der^m\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right) = Der\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right)$ for all $m \geq 2$. We set π the projection mapping of the tangent bundle TM to M, $\mathcal{N}_\Gamma$ and $\mathcal{N}_0$ are respectively the normalizer of $\mathfrak{A}_\Gamma^h$, of A_2. The first Chevalley-Eilenberg's cohomology of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is respectively $\left(\pi_*\left(\mathcal{N}_\Gamma\right)/\pi_*\left(\mathfrak{A}_\Gamma^h\right)\right) \otimes \left(\mathrm{End}\left(A_1\right)/adj_{\mathfrak{A}_\Gamma^v} \oplus \left(\left(\mathcal{N}_0 \ominus \mathfrak{A}_\Gamma^h\right)/A_2\right) \oplus \left(\mathrm{H}_R^1\left(\mathrm{B}\right) \otimes \mathbb{R}\right)^k\right)$ if $\mathfrak{A}_\Gamma^h$ is non-null everywhere and $\mathrm{End}\left(A_1\right)/adj_{\mathfrak{A}_\Gamma^v} \oplus \left(\left(\mathcal{N}_0 \ominus \mathfrak{A}_\Gamma^h\right)/A_2\right) \oplus \left(\mathrm{H}_R^1\left(\mathrm{B}\right) \otimes \mathbb{R}\right)^k$ if $\mathfrak{A}_\Gamma^h = \{0\}$ where $\mathrm{H}_R^1\left(\mathrm{B}\right)$ the first de Rham's cohomology on closed 1-differential forms relative to the foliation yielded by Hermann theorem, from A_2 towards $F_1\left(\mathcal{T}M\right)$ which is the base ring of $\mathfrak{A}_\Gamma^v$. We can tell that the derivative ideal $\left[\mathfrak{A}_\Gamma \cap \mathfrak{N}_R, \mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right]$ of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ coincides with itself and we find its normalizer. The following fact $\left[\mathfrak{A}_\Gamma \cap \mathfrak{N}_R, \mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right]$ equals to $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ leads to the stabilization of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ by all m-derivations of $\mathfrak{A}_\Gamma$, then another Lie algebra has been found which is $\mathfrak{A}_\Gamma/\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ composed by the corresponding classes of elements of $\mathfrak{A}_\Gamma \ominus \left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right)$. Similar results to those of $(m \geq 2)$-derivations of the module $\mathfrak{A}_\Gamma^v$ will be obtained for the $(m \geq 2)$-derivations of $\mathfrak{A}_{H^0}$. Moreover, the nullity space $\mathfrak{N}_R$ doesn't be a Lie algebra in general and in [11] we have given a sufficient condition for this space to be one. In this paper, we give another for Γ homogeneous of degree -1 and for Γ homogeneous of degree 0 or linear connection in the sense of Grifone. Particularly, $\mathfrak{N}_R$ is always a Lie algebra in the case of homogeneous of degree -1 connection. In the same goal, we discover that $\mathfrak{A}_\Gamma + \mathfrak{N}_R^h$, $\mathfrak{A}_\Gamma$ added to the vertical space are Lie algebras where the first is a Lie sub-algebra of the normalizer of $\mathfrak{N}_R^h$, and we find a necessary and sufficient condition for $\mathfrak{A}_\Gamma + \mathfrak{N}_R$ to be a Lie algebra. An example is given when Γ is not homogeneous where $\mathfrak{N}_R$ is a not a Lie algebra.

The second section treats the partial differential equations defining elements of $\mathfrak{A}_\Gamma$, of the common part of $\mathfrak{A}_\Gamma$ with the nullity space $\mathfrak{N}_R$, that is to say $\mathfrak{A}_\Gamma^h$, $\mathfrak{A}_\Gamma^v$ and of course the elements of $\mathfrak{A}_\Gamma$ which are outside of $\mathfrak{N}_R$. In [11], we have formulated one system of $4n^2$ partial differential equations which defines the elements of a Lie algebra vector fields on TM whose the Lie derivatives vanish an arbitrary 1-vector form on TM by the one in [7]. Unfortunately, there is one sign error in the formula, it doesn't influence the results of [11] because the arguments used in doesn't base on this system. Here, we sort a correct new corresponding formula and we considerably reduce the number of equations for a connection Γ. In local coordinates, we obtain one system of n^2 partial differential equations of first order (PDEFO) in the local components of the elements of $\mathfrak{A}_\Gamma$ knowing that they are all projectables. We deduce the (PDEFO) in the local components of the elements of $\mathfrak{A}_\Gamma^h$ and that of $\mathfrak{A}_\Gamma^v$. These results give us several precise ideas about local forms of the elements of these last two Lie algebras and those of $\mathfrak{A}_\Gamma/\left(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R\right)$. In [7], a Lie algebra of vector fields has been put in evidence, it is the Lie algebra of infinitesimal automorphisms $\overline{\mathfrak{A}_\Gamma}$ of a Grifone's connection of the form $\Gamma = [J, S]$ with S a semi-spray in $\mathcal{T}M$ where $JS = C$ (C is the Liouville vector field on TM), this connection is of weak torsion free cf. [3]. This Lie algebra is of finite dimension for some homogeneous condition on S. In this article, we prove that for a connection not necessarily of weak torsion free where Γ is homogeneous of degree different to 0, $\overline{\mathfrak{A}_\Gamma}$ is locally a Lie sub-algebra of affine vector fields on $\mathbb{R}^{2n}$ of dimension at most $n^2 + n$ from the above results on (PDEFO). In the case where locally all constant vector fields and the Euler vector field are present in the image of $\overline{\mathfrak{A}_\Gamma}$ by the inverse mapping relative to the complete lift on TM, all m-derivations of $\overline{\mathfrak{A}_\Gamma}$ are Lie derivatives with respect to elements of its normalizer $\mathcal{N}$. Thus, each m-derivation of $\mathcal{N}$ is inner. If the maximum dimension is everywhere attained, then every m-derivations of $\overline{\mathfrak{A}_\Gamma}$ are inner. The corresponding results about the first space of cohomology of Chevalley-Eilenberg will be given. We pay attention to the existence of an homogeneous of degree 0 connection where $\overline{\mathfrak{A}_\Gamma}$ is not a Lie sub-algebra of the Lie algebra of the locally affine vector fields. In the case where all above hypothesis are not satisfied, there are m-derivations which don't be Lie derivatives with respect to vector fields. The examples illustrating all our results in section 1 and 2 are located in the last part of this section 2. Several calculus in these examples use Maple software because of hardness of resolution of the (PDEFO) by calculation by hand.

The last section focus on global vision of the set of all connections in the smooth manifold M. Here, we express again that a Grifone's connection is more easy for searching algebraic structures than usual linear connections. Such as we have seen several Lie algebras generated by Grifone's connections in [11] through properties of Frölicher-Nijenhuis bracket on vector forms [2] used to calculations around the connection and its curvature. In this paper, we state that the arithmetic mean and weighted average, the composition of odd number of connections of Grifone on M is a connection and we put the Lie algebra $\mathfrak{A}_\Gamma$ and the curvature R of the resulting connection Γ with its nullity space $\mathfrak{N}_R$ based on those of connections in the calculations. These facts are not possible for the set of usual linear connections on M. Let K be the set of all Grifone's connections on M, one larger set than K containing more 1-vector forms on TM such as the identity mapping, the null mapping and the tangent structure J on TM forms a solvable of order 2 Lie algebra spanned by K from addition, mapping composition and outer multiplication of a mapping by real numbers. Our proofs use several results in the previous sections.

2. Results on Lie algebras of a connection by global methods

Definition 2.1. We recall one notion in order to define tangent structure J of TM in [11]. Let be the exact sequence of vector bundles on TM

$$0 \to \pi^*(TM) \xrightarrow{i} TTM \xrightarrow{j} \pi^*(TM) \to 0$$

with $\pi : TM \to M$ the projection of the tangent bundle to M; $P : TTM \to TM$ the projection of the tangent bundle to TM, i: the natural injection; $j = (P, \pi_*)$ where π_* is the linear tangent mapping of π. Thus, J equals to $i \circ j$ and locally in coordinates $(x^i, y^j)_{1\leq i,j\leq n}$ of TM, $J = dx^i \otimes \frac{\partial}{\partial y^i}$.

Definition 2.2. We say that a 1-vector form Γ of TM, C^∞ on $TM - \{0\} = \mathcal{T}M$ such that $J\Gamma = J$, $\Gamma J = -J$ is a Grifone's connection on M cf.[3].

By a theorem in [3], $\Gamma^2 = \mathrm{I}$ with I the identity mapping of $\chi(TM)$ (the Lie algebra of vector fields of TM). The connection Γ has two constant eigenvalues 1 and -1. The 1-vector form Γ is then an almost product on TM. The corresponding projectors are: the horizontal projector (resp. the vertical projector) is $h = \frac{1}{2}(\mathrm{I} + \Gamma)$ (resp. $v = \frac{1}{2}(\mathrm{I} - \Gamma)$).
We know that Γ equips M with a splitting of TTM (the tangent bundle of TM) into direct sum of horizontal and vertical spaces: $TTM = H(TM) \oplus V(TM)$ where

$$H(TM) = Im(h) = Ker(v) \text{ and } V(TM) = Im(v) = ker(h).$$

In local coordinates $(x^i, y^i)_{1\leq i,j\leq n}$ of TM, the connection Γ is $dx^i \otimes \frac{\partial}{\partial x^i} - 2\Gamma_i^j dx^i \otimes \otimes \frac{\partial}{\partial y^j} - dy^i \otimes \frac{\partial}{\partial y^i}$ cf. [3].
In the following, Γ will be a connection of Grifone.

Definition 2.3. The following 2-vector form $R = -\frac{1}{2}[h, h]$ where

$$\frac{1}{2}[h, h](X, Y) = [hX, hY] + h[X, Y] - h[hX, Y] - h[X, hY], \ \forall\ X, Y \in \chi(\mathcal{T}M), \tag{2.1}$$

with $\chi(\mathcal{T}M)$ the Lie algebra of vector fields on $\mathcal{T}M$, is called curvature of Γ. This curvature R is $t\,[\Gamma, \Gamma]$ where t is a constant. One vector space relative to R is the nullity space of curvature of Γ that is

$$\mathfrak{N}_R = \{X \in \chi(\mathcal{T}M) \text{ where } R(X,Y) = 0, \forall\, Y \in \chi(\mathcal{T}M)\}\,.$$

A distribution on a smooth manifold is a $F(M)$-sub-module of the module of all vector fields on M. In general, the space $\mathfrak{N}_R$ forms a non-involutive distribution on $\mathcal{T}M$, that is to say $\mathfrak{N}_R$ doesn't be a Lie algebra, however the horizontal nullity space $\mathfrak{N}_R^h = \{X \in \mathfrak{N}_R \text{ such that } h(X) = X\}$ is involutive cf. [11]. The mapping R is semi-basic (in other words if one of its arguments is vertical, the result becomes null and the image of R is in vertical space), then the vertical space is contained in $\mathfrak{N}_R$. The following proposition discuss conditions so that $\mathfrak{N}_R$ would be a Lie algebra (involutive), different to the results of [11]. About the curvature R in term of local coordinates $(x^i, y^j)_{1\leq i,j\leq n}$ of TM:

$$R = \frac{1}{2}R^i_{jk}dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i} \text{ where } R^i_{jk} = \frac{\partial \Gamma^i_k}{\partial x^j} - \frac{\partial \Gamma^i_j}{\partial x^k} + \Gamma^l_k \frac{\partial \Gamma^i_j}{\partial y^l} - \Gamma^l_j \frac{\partial \Gamma^i_k}{\partial y^l}. \tag{2.2}$$

The ring of real smooth functions on M resp. on TM and on $\mathcal{T}M$ is denoted by $F(M)$ resp. by $F(TM)$ and $F(\mathcal{T}M)$.

Proposition 2.4. *If the connection is homogeneous of degree -1 ($[C, \Gamma] = -\Gamma$), then the nullity space of its curvature is involutive. In the case where the connection is homogeneous of degree 0 ($[C, \Gamma] = 0$), that is to say a linear connection Γ in the sense of Grifone, $\mathfrak{N}_R$ is involutive if and only if $\mathfrak{N}_R^h$ without its part into the vertical space is included in the nullity space of the corresponding linear connection on M.*

Proof. As we know, the vertical space $v\left(\chi\left(\mathcal{T}M\right)\right)$ is involutive, the same for $\mathfrak{N}_R^h$, then $\mathfrak{N}_R = \mathfrak{N}_R^h \oplus v\left(\chi\left(\mathcal{T}M\right)\right)$ is an involutive distribution if and only if cf. [11]

$$\left[\mathfrak{N}_R^h, v\left(\chi\left(\mathcal{T}M\right)\right)\right] \subset \mathfrak{N}_R.$$

We reason locally, so in local chart of the system $(x^i, y^j)_{1\leq i,j\leq n}$. As $v\left(\chi\left(\mathcal{T}M\right)\right)$ is a $F\left(\mathcal{T}M\right)$-module, then it is sufficient to take $X = X^j \frac{\partial}{\partial x^j} \in \mathfrak{N}_R$ and $\frac{\partial}{\partial y^i}$ for fixed i and we do $\left[X, \frac{\partial}{\partial y^i}\right]$ for all i in order to obtain the result. This bracket belongs to $\mathfrak{N}_R$ if and only if $\frac{\partial X^j}{\partial y^i}\frac{\partial}{\partial x^j} \in \mathfrak{N}_R$. But $X \in \mathfrak{N}_R$, then $X^j R^u_{jt} = 0$ for all t, u. Deriving with respect to y^i this last relation and considering the homogeneity of order -1 of Γ (ie the local form of R doesn't depend on y^i), we have $\frac{\partial X^j}{\partial y^i}\frac{\partial}{\partial x^j} \in \mathfrak{N}_R$ if and only if

$$X^j \frac{\partial R^u_{jt}}{\partial y^i} = 0 \text{ for all } i, t, u. \tag{2.3}$$

Thus, we obtain the first assertion. Next, we know that a linear connection of Grifone is homogeneous of degree 0 where its coefficients are defined by $\Gamma^t_j = y^s \gamma^t_{js}(x^g, 1 \leq g \leq n)$ and γ^t_{js} the symbols of Christoffel of the corresponding linear connection on M. For such connection

$$R = \frac{1}{2}y^l R^i_{ljk}dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i} \text{ where } R^i_{tjk} = \frac{\partial \gamma^i_{kt}}{\partial x^j} - \frac{\partial \gamma^i_{jt}}{\partial x^k} + \gamma^l_{kt}\gamma^i_{jl} - \gamma^l_{jt}\gamma^i_{kl},$$

with

$$y^l R^i_{ljk}(x^g, 1 \leq g \leq n) = R^i_{jk}.$$

Going back to (2.3), we have another equivalent relation $\underset{1\leq t\leq n}{X^t R^u_{itk}} = 0$ for all $1 \leq i, k, u \leq n$. That is to say $X = X^j \frac{\partial}{\partial x^j}$ is in the usual nullity space of curvature of the linear connection. □

Remark 2.5. We can give an example of non-homogeneous connection such that the nullity space of the curvature is non-involutive. We will see it later in Example 3.16.

We say that a vector field X belongs to $\mathfrak{A}_Q$ with Q is 1-vector form on TM if $[X, Q] = 0$ or the Lie derivative $L_X Q = 0$ or for all $Y \in \chi\left(\mathcal{T}M\right)$, $Q\left[X, Y\right] = [X, Q(Y)]$. It is well known that the set of these X forms a Lie algebra. It's easy to find that $\mathfrak{A}_\Gamma = \mathfrak{A}_h = \mathfrak{A}_v$.

Furthermore, it is clear in this case that $[X, hX] = 0$ for $X \in \mathfrak{A}_\Gamma$. This $\mathfrak{A}_\Gamma$ is a Lie algebra and we take from two interesting Lie algebras that are ideals of $\mathfrak{A}_\Gamma$ cf. [11]:

$$\mathfrak{A}_\Gamma^h = \{X \in \mathfrak{A}_\Gamma \text{ such that } h(X) = X\} \text{ and } \mathfrak{A}_\Gamma^v = \{X \in \mathfrak{A}_\Gamma \text{ with } v(X) = X\}.$$

We recall that a vector field on TM is projectable if the linear tangent mapping of the canonical projection of the tangent bundle to M applied to this vector is a vector field on M.

Proposition 2.6. *The set $\mathfrak{A}_\Gamma + \mathfrak{N}_R^h$ respectively $\mathfrak{A}_\Gamma + v(\chi(\mathcal{T}M))$ is a Lie sub-algebra of the normalizer of $\mathfrak{N}_R^h$ (resp. is a Lie algebra). If $\mathfrak{A}_\Gamma^h = \{0\}$, then the first sum becomes direct in term of module. The $\mathfrak{A}_\Gamma + \mathfrak{N}_R$ is a Lie algebra if and only if $\left[\mathfrak{N}_R^h, v(\chi(\mathcal{T}M))\right] \subset \mathfrak{N}_R$.*

Proof. We have $[\mathfrak{A}_\Gamma, \mathfrak{A}_\Gamma] \subset \mathfrak{A}_\Gamma$, $\left[\mathfrak{N}_R^h, \mathfrak{N}_R^h\right] \subset \mathfrak{N}_R^h$ and $[\mathfrak{N}_R^v, \mathfrak{N}_R^v] \subset \mathfrak{N}_R^v$ because $\mathfrak{N}_R^v = v(\chi(\mathcal{T}M))$. We obtain $[\mathfrak{A}_\Gamma, \mathfrak{N}_R]$ is included in $\mathfrak{N}_R$. Indeed, if $X \in \mathfrak{A}_\Gamma$ then $[X, \Gamma] = 0$. By Jacobi identity for vector forms, we have $[X, ([\Gamma, \Gamma] = 2R)] = 0$. We recall this identity [2]: for all vector forms K, L, V on M of respective degrees k, l, v, then

$$[K, [L, V]] = [[K, L], V] + (-1)^{kl} [L, [K, V]]. \tag{2.4}$$

So we get for all $Y, Z \in \chi(\mathcal{T}M)$:

$$[X, R(Y, Z)] - R([X, Y], Z) - R(Y, [X, Z]) = 0.$$

If we take $Y \in \mathfrak{N}_R$, we obtain $[X, Y] \in \mathfrak{N}_R$. Thus for $Y \in \mathfrak{N}_R^h$ where $hY = Y$, we have $[X, Y] \in h(\chi(\mathcal{T}M))$ and then in $\mathfrak{N}_R^h$. We simply replace h by v in this last reasoning in order to have $[X, Y] \in v(\chi(\mathcal{T}M))$ basing on the fact $\mathfrak{A}_\Gamma = \mathfrak{A}_v$. Moreover, we know that $\mathfrak{A}_\Gamma^h = H^0 \cap \mathfrak{N}_R$ by The Proposition 3.12 of [11]. Then, $\mathfrak{A}_\Gamma^h = \{0\}$ says $H^0 \cap \mathfrak{N}_R = \{0\}$. Consequently, $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R^h = \{0\}$ because all elements of $\mathfrak{A}_\Gamma$ are projectable cf. [11]. Thus the direct sum indicated above. The last affirmation is immediate by arguments in the first two sentences at the beginning of the present proof. □

It is known that $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R = \mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$, the study of $\mathfrak{A}_\Gamma^h$ and of $\mathfrak{A}_\Gamma^v$ started in [11]. We try to complete it and do more in the present paper. Furthermore, the Lie algebra $\mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$ is the measure of the obstruction of $\mathfrak{A}_\Gamma + \mathfrak{N}_R$ to be direct sum of modules. As for that of Lie algebra $\mathfrak{A}_\Gamma + \mathfrak{N}_R^h$ (resp. $\mathfrak{A}_\Gamma + v(\chi(\mathcal{T}M))$), it is $\mathfrak{A}_\Gamma^h$ (resp. $\mathfrak{A}_\Gamma^v$) which expresses this obstruction.
To do so, we recall some notions and results

Definition 2.7. The centralizer of a Lie algebra A in another $B \supset A$ is defined by $\{X \in B, [X, A] = \{0\}\}$. The center of A is got for replacing B by A.

Proposition 2.8. *([11]) The Lie algebra $\mathfrak{A}_\Gamma^v$ is the centralizer of the Lie algebra $\mathfrak{A}_{H^0}$ spanned by the set of all horizontal projectable vector fields H^0, where $\mathfrak{A}_{H^0}$ is a $F(M)$-module of $\chi(\mathcal{T}M)$. A vector field is in $\mathfrak{A}_\Gamma^v$ if and only if this vector commutes with all those of H^0.*

Definition 2.9. A generalized foliation $\mathfrak{F} = \{\mathfrak{F}^\alpha\}_{\alpha \in I}$ on M is a partition into connected sub-manifolds of $M = \bigcup_{\alpha \in I} \mathfrak{F}^\alpha$ such that $\mathfrak{F}^\alpha \cap \mathfrak{F}^\beta = \varnothing$ for $\alpha \neq \beta$, which are exactly the orbits of flow's composition generated by the vector fields locally tangent to the leaves $\mathfrak{F}^\alpha_{\alpha \in I}$ of $\mathfrak{F}$ cf. [12]. A foliation is singular if it exists a leaf of null dimension.

Theorem 2.10. *(R. Hermann [6]) All involutive distributions locally finitely generated are integrable.*

We consider a Lie algebra A^0 spanned by the set of all horizontal projectable vector fields as a $F(\mathcal{T}M)$-module. By the above theorem, A^0 is integrable because it is locally spanned by $\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}$ for $i = 1, \ldots, n$. It induces a generalized foliation on $\mathcal{T}M$. Of course, this foliation is non-singular because A^0 is everywhere non-null. As the Lie algebra $\mathfrak{A}_{H^0}$ is generated by the set of all horizontal projectable vector fields, it is locally spanned by $\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}$ for $i = 1, \ldots, n$ in term of $F(M)$-module. We have $\mathfrak{A}_{H^0} \subset A^0$. The set of regular points will be noted by Reg such that each point admits a neighborhood where $\mathfrak{A}^0$ is of non-null positive constant rank (the corresponding leaf is of constant dimension). This set Reg is dense in $\mathcal{T}M$ cf. [11]. Let $x \in \mathrm{Reg}$ and $\mathcal{T}U$ an open set of adapted chart of the foliation containing x where $\mathfrak{A}^0$ is of rank $n+k>0$ (the integer k can be zero). That is to say it exists a coordinates system $\left(x^i, y^j\right)_{1\leq i,j\leq n}$ (to simplify we use in here the same name of the usual coordinates system of $\mathcal{T}M$) where $\mathfrak{A}^0$ can be written as the set of:

$$X^i \underset{1\leq i\leq n}{\frac{\partial}{\partial x^i}} + X'^j \underset{1\leq j\leq k}{\frac{\partial}{\partial y^j}} \text{ where the } X^i, X'^j \in F\left(\mathcal{T}U\right). \tag{2.5}$$

We suppose in the following that $\mathfrak{A}_{H^0}$ is locally in the above adapted chart

$$X^i \underset{1\leq i\leq n}{\frac{\partial}{\partial x^i}} + X'^j \underset{1\leq j\leq k}{\frac{\partial}{\partial y^j}} \text{ where the } X^i, X'^j \in F\left(\mathcal{T}U\right). \tag{2.6}$$

In the following, $\mathcal{T}U$ is always an adapted chart with a coordinates system $\left(x^i, y^j\right)_{1\leq i,j\leq n}$. We consider $\mathrm{Reg} = \mathcal{T}M$ to simplify our reasoning (otherwise, we follow the results in Reg and can pass through limits to have those in $\mathcal{T}M$).
In consequence,

Proposition 2.11. *The elements of the Lie algebra $\mathfrak{A}_\Gamma^v$ on $\mathcal{T}U$ are of the form $X'^j \underset{1\leq j\leq n}{\frac{\partial}{\partial y^j}}$ where the $X'^j\left(y^t, k+1\leq t\leq n\right) \in F\left(\mathcal{T}U\right)$.*

Proof. We know that $\mathfrak{A}_\Gamma^v$ is the centralizer of $\mathfrak{A}_{H^0}$, then we use the definition of Lie algebra centralizers. So, let $X \in \mathfrak{A}_\Gamma^v$ in the form $X^i \underset{1\leq i\leq n}{\frac{\partial}{\partial x^i}} + X'^j \underset{1\leq j\leq n}{\frac{\partial}{\partial y^j}}$ where the $X^i, X'^j \in F\left(\mathcal{T}U\right)$. By definition and using the fact that $\mathfrak{A}_{H^0}$ is a $F\left(U\right)$-module, we obtain $X^i = 0$ for all i. In the same way, the X'^j depend only on $\left(y^t, k+1\leq t\leq n\right)$. □

Remark 2.12. The Proposition 3.4 of [11] saying that $\mathfrak{A}_\Gamma^v$ is the common part of the centralizer of $\mathfrak{A}_\Gamma^h$ with $\mathfrak{A}_\Gamma$, doesn't explicit the form of elements of $\mathfrak{A}_\Gamma^v$ because of restrictive condition "$\mathfrak{A}_\Gamma^h$ is everywhere non-null" and the form of elements of $\mathfrak{A}_\Gamma$ is badly-known. Contrary to $\mathfrak{A}_\Gamma^h$, we remark that $\mathfrak{A}_\Gamma^v$ is never reduced to zero as one example in the following will show. In fact, the sum between $\mathfrak{A}_\Gamma$ and $v\left(\chi\left(\mathcal{T}M\right)\right)$ cf. the Proposition 2.6 is never direct. Similarly, $\mathfrak{A}_\Gamma + \mathfrak{N}_R$ is not a direct sum of modules.

We assign that:

Definition 2.13. The derivative ideal of a Lie algebra L denoted by $[L, L]$ is a Lie algebra generated by the $[X, Y]; X, Y \in L$. By recurrence, we set $\mathrm{D}^1(L) = [L, L], \mathrm{D}^u = D^1\left(\mathrm{D}^{u-1}\right)$ for $u > 1$. The algebra L is solvable of order $u > 1$ if for all $h < u$, $\mathrm{D}^h(L) \neq \{0\}$ and $\mathrm{D}^u(L) = \{0\}$.

Theorem 2.14. *On $\mathcal{T}U$, $\mathfrak{A}_\Gamma^v$ is a semi-direct sum of Lie algebras A_1 and A_2 where $[A_1, A_1] = \{0\}$, $[A_1, A_2] = A_1$, $[A_2, A_2] = A_2$ and we get $[\mathfrak{A}_\Gamma^v, \mathfrak{A}_\Gamma^v] = \mathfrak{A}_\Gamma^v$.*

Proof. The element X of $\mathfrak{A}_\Gamma^v$ is well-known by Proposition 2.11. Thus, we split these X in the following way:

$$X'^i \underset{1\leq i\leq k}{\frac{\partial}{\partial y^i}} + X'^j \underset{k+1\leq j\leq n}{\frac{\partial}{\partial y^j}} \text{ where the } X'^u\left(y^t, k+1\leq t\leq n\right) \in F\left(\mathcal{T}U\right).$$

We set $A_1\left(\mathcal{T}U\right)$ the set of the first vector fields and $A_2\left(\mathcal{T}U\right)$ that of the second, corresponding to the splitting. Moreover, we designate by $F_1\left(\mathcal{T}U\right)$ the base ring of $\mathfrak{A}_\Gamma^v\left(\mathcal{T}U\right)$. It is clear that $A_1\left(\mathcal{T}U\right)$ is commutative by usual bracket of vector fields. We can also say that $\mathfrak{A}_\Gamma^v\left(\mathcal{T}U\right)$ is both a semi-direct sum of Lie algebras and direct sum of modules. For the following affirmation, it is sufficient to chose good monomials as components of the vector fields of $A_2\left(\mathcal{T}U\right)$. For the next assertion, we use the arguments of [11] p.699. We deduce that $\left[\mathfrak{A}_\Gamma^v\left(\mathcal{T}U\right), \mathfrak{A}_\Gamma^v\left(\mathcal{T}U\right)\right] = \mathfrak{A}_\Gamma^v\left(\mathcal{T}U\right)$. We can define globally the Lie algebras A_1 and A_2 and we achieve our proof. □

We have a similar theorem to that of [11] based on H^0:

Proposition 2.15. *The Lie algebra $\mathfrak{A}_{H^0}$ coincides with H^0 if and only if $R = 0$.*

Proof. By (2.1) and the fact that the curvature is semi-basic, the horizontal space is involutive if and only if the connection is flat. Then, we get the result knowing that H^0 is locally spanned by all local elements of natural base of horizontal space on $F(M)$. □

Then we obtain

Corollary 2.16. *The integer k equals to zero if and only if $R_{|\mathcal{T}U} = 0$.*

Proof. The proof bases on the dimension of $\mathfrak{A}_{H^0}$ on $\mathcal{T}U$, it is equal to n which is the dimension of H^0 if and only if $k = 0$. By our last proposition, we get the equivalence. □

If $k = 0$ in $\mathcal{T}U$, then

$$(A_1)_{\mathcal{T}U} = \{0\} \text{ and } (A_2)_{\mathcal{T}U} = \left\{ X'^j \left(y^t, 1 \le t \le n\right) \frac{\partial}{\partial y^j} \right\}_{1\le j\le n}.$$

In the following, we suppose that $R \neq 0$ unless expressed mention.

Corollary 2.17. *The derivative ideal of the Lie algebra $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ itself.*

Proof. The Lie algebra $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R = \mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$ by Proposition 3.10 of [11] and $\left[\mathfrak{A}_\Gamma^h, \mathfrak{A}_\Gamma^h\right] = \mathfrak{A}_\Gamma^h$ by Theorem 3.20 of [11]. We can affirm that $[\mathfrak{A}_\Gamma^v, \mathfrak{A}_\Gamma^v] = \mathfrak{A}_\Gamma^v$ following the Theorem 2.14. Thus,

$$[\mathfrak{A}_\Gamma \cap \mathfrak{N}_R, \mathfrak{A}_\Gamma \cap \mathfrak{N}_R] = \mathfrak{A}_\Gamma \cap \mathfrak{N}_R.$$

□

We suppose that $m \ge 2$ is an natural integer in the following.

Definition 2.18. A mapping $\mathbb{R}-$linear D from $\mathfrak{A}$ into $\mathfrak{A}$ is a m-derivation of a Lie algebra $\mathfrak{A}$ if $\forall X_1, X_2, \dots, X_m \in \mathfrak{A}$,

$$\begin{aligned} D\left[X_1, [X_2, \dots, [X_{m-1}, X_m] \dots]\right] &= \left[D(X_1), [X_2, \dots [X_{m-1}, X_m] \dots]\right] + \\ &\quad + \left[X_1, [D(X_2), \dots, [X_{m-1}, X_m] \dots]\right] + \\ &\quad + \cdots + \\ &\quad + \left[X_1, [X_2, \dots, [D(X_{m-1}), X_m] \dots]\right] + \\ &\quad + \left[X_1, [X_2, \dots, [X_{m-1}, D(X_m)] \dots]\right]. \end{aligned} \tag{2.7}$$

In $m = 2$, this definition coincides with that of derivation. Actually, all derivations are m-derivations for all $m \ge 2$ but the converse is false as Theorem 3.10 of [10] shows, that is why the study of these m-derivations with $m > 2$ is interesting.

Definition 2.19. Let $\mathfrak{A}$ be a Lie algebra, we define $\mathfrak{C}^1(\mathfrak{A}) = \mathfrak{A}$ and for all $u > 1$, $\mathfrak{C}^u(\mathfrak{A}) = \left[\mathfrak{A}, \mathfrak{C}^{u-1}(\mathfrak{A})\right]$ cf.[1]. The Lie algebra $\mathfrak{A}$ is nilpotent of order $u > 1$ if u is the smaller integer such that $\mathfrak{C}^u(\mathfrak{A}) = \{0\}$.

By the last corollary, the Lie algebra $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is never solvable. Following [1], this Lie algebra is never nilpotent. Then, we discuss its m-derivations which are not trivial.

Proposition 2.20. *For $m \ge 3$, all linear mappings from A_1 towards $\mathfrak{A}_\Gamma^v$ are m-derivations.*

Proof. When $m \ge 3$, the relation (2.7) is trivially verified by all linear mapping from A_1 to $\mathfrak{A}_\Gamma^v$ because of the following facts $[A_1, A_2] = A_1$ and $[A_1, A_1] = \{0\}$. Thus, we achieve our proof. □

Remark 2.21. The relation (2.7) of Proposition 2.20 is not obvious when $m = 2$.

We recall that a mapping f from one part of $\chi(M)$ to another is local if for all $X \in \chi(M)$ and an open V of M such that $X_{|V} \equiv 0$, we have $f(X)_{|V} \equiv 0$.

Proposition 2.22. *All derivations from A_1 to A_2 are null.*

Proof. If D is such derivation, then for X_1, X_2 in A_1, we have $D[X_1, X_2] = [DX_1, X_2] + [X_1, DX_2]$. As $[A_1, A_1] = \{0\}$, we can obtain

$$[DX_1, X_2] = -[X_1, DX_2].$$

We will verify that D is local, that is to say if X_1 on A_1 vanishes on an open set of $\mathcal{T}M$, then $D(X_1)$ is null on this open (we can take an adapted chart). Let X_1 verifying such hypotheses on the open set $\mathcal{T}U$ with $k \neq 0$, with the previous relation:

$$\left[(DX_1)_{|\mathcal{T}U}, X_{2|\mathcal{T}U}\right] = 0, \text{ for all } X_2 \in A_1. \tag{2.8}$$

If i_1 an integer between $k+1$ and n, and i_0 another one between 1 and k, this last relation leads in $X_2 = y^{i_1}\frac{\partial}{\partial y^{i_0}}$ that

$$(DX_1)^{i_1}_{|\mathcal{T}U}\frac{\partial}{\partial y^{i_0}} = 0,$$

and it's for all i_1. Then $(DX_1)_{|\mathcal{T}U} = 0$ and we get locality of D. In this case, on the adapted chart $\mathcal{T}U$, $D = \underset{k+1\leq i\leq n}{w^i \otimes \frac{\partial}{\partial y^i}}$ with $w^i = \underset{1\leq j\leq k}{w^i_j dy^j}$ and the $w^i_j \in F(\mathcal{T}U)$. For $X_1 = \frac{\partial}{\partial y^{j_0}}$ and $X_2 = y^{i_0}\frac{\partial}{\partial y^{t_0}}$ with j_0 between 1 and k, i_0 between $k+1$ and n, t_0 between 1 and k, we obtain from (2.8) that each $w^{i_0}_{j_0} = 0$ and it is for all i_0, j_0. Thus, each $w^i = 0$ and $D_{|\mathcal{T}U} = 0$ for all $\mathcal{T}U$ where $k \neq 0$. The case of $\mathcal{T}U$ with $k = 0$ is trivial. So D is identically null. $\square$

Proposition 2.23. *A m-derivation of $\mathfrak{A}^v_\Gamma$ is split into $D = D^{11} + D^{12} + D^{21} + D^{22}$ where D^{11} is a linear part of D from A_1 to A_1, D^{12} from A_1 into A_2, D^{21} from A_2 towards A_1 and D^{22} from A_2 to A_2. Thus, all D^{ij} are m-derivations.*

Proof. It is clear that for all $m \geq 2$, D^{11} is always a m-derivation because (2.7) is trivially satisfied for this mapping. By Proposition 2.20, D^{12} is equally a $(m \geq 3)$-derivation. So $D^{21} + D^{22}$ is one. Now, we consider $X_1, X_2, \ldots, X_m \in A_2$ and we have the equality (2.7) corresponding to $D^{21} + D^{22}$. Using $[A_1, A_2] = A_1$, $[A_2, A_2] = A_2$ and the module's direct sum $A_1 \oplus A_2$ in the previous equality, D^{21} and D^{22} are $(m \geq 3)$-derivations. For $m = 2$, it is sufficient to check that the last three D^{ij} are derivations. We proceed as in the previous knowing $D^{12} = 0$ cf. Proposition 2.22 and acting $D^{21} + D^{22}$ on X_1, X_2 of A_2. It yields D^{21}, D^{22} to be derivations. It achieve our proof. $\square$

Proposition 2.24. *All m-derivations from A_2 towards A_1 are local.*

Proof. Suppose that D such m-derivation and $X \in A_2$ is null on a open set $\mathcal{T}U$ where U is an open of M (we can consider as a adapted chart with $k \neq 0$, the other $k = 0$ is trivial). We reason by contradiction, that is $D(X)_{|\mathcal{T}U} \neq 0$. It exists $x \in \mathcal{T}U$ and a neighborhood $V_x \subset \mathcal{T}U$ of x such that for all $z \in V_x$, $D^{i_0}(X)(z) \neq 0$ (i_0-th component of $D(X)(z)$) with i_0 between 1 and k. As in [14] pp.166-167, we have fabricated a plateau function on M corresponding to a sub-ring of $F(M)$, then we can have one adapted to the base ring of A_2. Thus, it is $f \in F_1(\mathcal{T}U)$ with $f_{|V_x} = y^{i_0}$ where $Supp(f) \subset \mathcal{T}U$ and $Y_{m-1}, \ldots, Y_2, Y_1 \in A_2$ such that $Y_{m-1} = y^{j_0}\frac{\partial}{\partial y^{j_0}}, Y_{m-2} = y^{j_0}\frac{\partial}{\partial y^{j_0}}, \ldots, Y_2 = y^{j_0}\frac{\partial}{\partial y^{j_0}}, Y_1 = \frac{\partial}{\partial y^{j_0}}$ in order that $[X, [Y_2, \ldots [Y_{m-1}, fY_1] \ldots]] = 0$. So (2.7) for $D[X, [Y_2, \ldots [Y_{m-1}, fY_1] \ldots]]$ induces to a contradiction on V_z. Then D is local. $\square$

We look back at two propositions of [14] assembled to one proposition hereafter. A system S is a set of q vector fields of rank p which commutes mutually and yields on a smooth manifold of dimension $n+q$ the Lie algebra A_S. There's a foliation generated by S cf. [14]. This algebra A_S coincides with the $F_0(U')$-module of vector fields on U' (with U' an adapted chart of this foliation) spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m+q-p}}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^p}$ where $p \geq 1$ and $F_0(U')$ the ring of real functions smooth on U' depending only on $x^1, \ldots, x^{m+q-p}$. The module $A_S(U')$ can be written as semi-direct product

$$A_S(U') = A^1_S(U') \oplus A^2_S(U')$$

where $A^1_S(U')$ is the sub-algebra of $A_S(U')$ generated by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m+q-p}}$ on the ring $F_0(U')$ and $A^2_S(U')$ is commutative ideal of $A_S(U')$ spanned by $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^p}$ on $F_0(U')$.

An inner m-derivation of a Lie algebra A with respect to an element of a Lie algebra B is a Lie derivative with respect to this element. It is inner if this element is in A.

Proposition 2.25. *Let D be a local derivation of A_S to A_S^2. It exists closed $1-$differential forms α^i and ω^i in U', with $i = 1, \ldots, p$ such that:*

(1) $D_{U'} = \left(\alpha^j + \omega^j\right) \otimes \frac{\partial}{\partial y^j}$, where each $ker(\alpha^j)$ contains $A_S^2(U')$ and each $ker(\omega^j)$ contains $A_S^1(U')$.

(2) all $\alpha^i\left[X,Y\right] = X.\alpha^i\left(Y\right) - Y.\alpha^i\left(X\right)$, for all vector fields $X, Y \in A_S(U')$.

We note $D_{U'}^{\alpha,\omega}$ the derivation $\left(\alpha^j + \omega^j\right) \otimes \frac{\partial}{\partial y^j}$ of $A_S\left(U'\right)$ towards $A_S^2\left(U'\right)$. The derivation $D_{U'}^{\alpha,\omega}$ of $A_S\left(U'\right)$ to $A_S^2\left(U'\right)$ with $\alpha = \left(\alpha^1, \ldots, \alpha^p\right)$ and $\omega = \left(\omega^1, \ldots, \omega^p\right)$ is an inner derivation if and only if, for all i, $\omega^i \equiv 0$ and α^i are exact. In that case we get $D_{U'}^{\alpha,\omega} = -\mathrm{L}_{f^i \frac{\partial}{\partial y^i}}$ where each $\alpha^i = df^i$ with f^i are functions in $F_0(U')$. Conversely, all $D_{U'}^{\alpha,\omega}$ verifying (1), (2) are derivations of A_S towards A_S^2.

Proposition 2.26. *Each m-derivation D of A_2 into A_1 is a sum of Lie derivative with respect to an element of A_1 and a non inner derivation of the form $D^{\alpha,0}$ where α is a non-exact closed 1-differential form from A_2 towards $F_1\left(\mathcal{T}M\right)$.*

Proof. Sketch of the demonstration is the following. We remark that Proposition 2.25 can be adapted to a derivation (2-derivation) from A_2 to A_1, where we replace $F_0\left(U'\right)$ by $F_1\left(\mathcal{T}U\right)$ where $k \neq 0$ takes the place of p, $A_S^2\left(U'\right)$ by $A_1\left(\mathcal{T}U\right)$, $A_S^1\left(U'\right)$ by $A_2\left(\mathcal{T}U\right)$. Then ω^j are null and the $\alpha^i = \underset{k+1 \leq j \leq n}{\alpha_j^i dy^j}$ with $1 \leq i \leq k$ are the following $\underset{k+1 \leq t,j \leq n}{\left(\frac{\partial \alpha_j^i}{\partial y^t} - \frac{\partial \alpha_t^i}{\partial y^j}\right)} = 0$ by (2), of course the $\underset{1 \leq t \leq k}{\frac{\partial \alpha_j^i}{\partial y^t}} = 0$ and $\underset{1 \leq t \leq n}{\frac{\partial \alpha_j^i}{\partial x^t}} = 0$. The expression "the α^i is exact" means it exists $f^i \in F_1\left(\mathcal{T}U\right)$ such that $\alpha^i = df^i$. For a $(m > 2)$-derivation D which is local by Proposition 2.24, we have on the adapted chart $\mathcal{T}U$, $D_{\mathcal{T}U} = \alpha^j \otimes \underset{1 \leq j \leq k}{\frac{\partial}{\partial y^j}}$ with α^j a 1-differential form of $A_2\left(\mathcal{T}U\right)$ towards $F_1\left(\mathcal{T}U\right)$. We set the obtained equality from

$$D_{\mathcal{T}U}\left[\frac{\partial}{\partial y^{a_0}}, \left[y^a \frac{\partial}{\partial y^a}, \ldots, \left[y^a \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^{b_0}}\right] \cdots\right]\right]$$

in $(0, \ldots, 0)$ for the a, b, a_0, b_0 between $k+1$ et n. Then we know

$$\left(\frac{\partial \alpha_{a_0}^i}{\partial y^{b_0}} - \frac{\partial \alpha_{b_0}^i}{\partial y^{a_0}}\right)(0, \ldots, 0) = 0,$$

for all a_0, b_0 and we get a more general equality by coordinates translations. It is clear that $\frac{\partial \alpha_{a_0}^i}{\partial y^c} = 0$ for $1 \leq c \leq k$ and $\frac{\partial \alpha_{a_0}^i}{\partial x^t} = 0$ for $1 \leq t \leq n$. These facts conduct to the closeness of α^i and of α. Then we have (2) and we achieve the demonstration of the Proposition 2.25 adapted to a m-derivation of $A_2\left(\mathcal{T}U\right)$ to $A_1\left(\mathcal{T}U\right)$ proceeding as in the previous one for $k \neq 0$ and to the triviality for $k = 0$. We demonstrate that the $D^{\alpha^i,0}$ for α 1-differential form of $A_2\left(\mathcal{T}U\right)$ to $F_1\left(\mathcal{T}U\right)$ are always derivations because the α^i are closed. All results of the present proof is true for each adapted chart $\mathcal{T}U$, thus by the property of the foliation, we get all these results on $\mathcal{T}M$. $\square$

Proposition 2.27. *Each m-derivation of A_2 is a Lie derivative with respect to its normalizer.*

Proof. Remind us that the normalizer of a Lie algebra A in another one $B \supset A$ is defined by $\{X \in B, [X, A] \subset A\}$. We proceed as in Proposition 2.24 in order that all m-derivations of A_2 are local. Next, we can adapt the proof of Theorem 2.1 of [13] taking into account the presence of all functions needed in this proof in the base ring of the module A_2. $\square$

Theorem 2.28. *We suppose $m > 2$. When $\mathfrak{A}_\Gamma^h$ is everywhere non-null, all m-derivations of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ are sum of a Lie derivative with respect to a vector field on the normalizer $\mathcal{N}_\Gamma$ of $\mathfrak{A}_\Gamma^h$ and a m-derivation of $\mathfrak{A}_\Gamma^v$ that is to say sum of endomorphism of A_1, of a linear* *mapping from A_1 to A_2, of a Lie derivative with respect to an element of A_1, of a derivation in the form $D^{\alpha,0}$ of Proposition 2.26 and of a Lie derivative with respect to a vector field in the normalizer $\mathcal{N}_0$ to A_2 on $\chi(\mathcal{T}M)$. On the contrary if $\mathfrak{A}_\Gamma^h = \{0\}$, then all m-derivations of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ are those of $\mathfrak{A}_\Gamma^v$ as cited above. For $m = 2$, the result for the derivations is composed by the same sentences as for $(m > 2)$-derivations without only one "of a linear mapping from A_1 to A_2". If the connection is flat then each $(m \geq 2)$-derivation of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is inner.*

Proof. Suppose that $m > 2$, we can affirm that all m-derivations of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ are direct sum of m-derivations of $\mathfrak{A}_\Gamma^h$ with another one of $\mathfrak{A}_\Gamma^v$ by using the same arguments as in [9] taking into account $[\mathfrak{A}_\Gamma^v, \mathfrak{A}_\Gamma^v] = \mathfrak{A}_\Gamma^v$ from Theorem 2.14 on $\mathcal{T}M$ and $\left[\mathfrak{A}_\Gamma^h, \mathfrak{A}_\Gamma^h\right] = \mathfrak{A}_\Gamma^h$ of [11]. If $\mathfrak{A}_\Gamma^h$ is non-vanishing everywhere, we know that m-derivations de $\mathfrak{A}_\Gamma^h$ are Lie derivatives with respect to elements of its normalizer $\mathcal{N}_\Gamma$ cf. [13] p.5. Furthermore, let D be a m-derivation of $\mathfrak{A}_\Gamma^v = A_1 + A_2$. It can be split into $D = D^{11} + D^{12} + D^{21} + D^{22}$ where D^{11} the linear part of D from A_1 to A_1, D^{12} of A_1 to A_2, D^{21} from A_2 towards A_1 and D^{22} of A_2 towards A_2. By Proposition 2.23, all these D^{ij} are m-derivations. Then the result comes from the Propositions 2.20, 2.23, 2.26 and 2.27. When $\mathfrak{A}_\Gamma^h$ is reduced to zero, the result is obtained by consideration for the above m-derivations of $\mathfrak{A}_\Gamma^v$. The one difference between derivations and $(m > 2)$-derivation is that from Proposition 2.22 saying $D^{12} = 0$ for derivations, it yields the next result.
If the connection is flat, $\mathfrak{A}_\Gamma^h$ is everywhere non-null. Then the m-derivations of $\mathfrak{A}_\Gamma^h$ are the same as in the previous. As $R = 0$ then $k = 0$ everywhere on $\mathcal{T}M$ by Corollary 2.16. In this case $A_1 = \{0\}$ and all m-derivations of $\mathfrak{A}_\Gamma^v$ are those of A_2, so they are Lie derivatives with respect to elements of $\mathcal{N}_0$. But Theorem 4.4 of [11] says that the normalizer of $\mathfrak{A}_\Gamma^v$ which is $\mathcal{N}_0$ and that of $\mathfrak{A}_\Gamma^h$ is $\mathcal{N}_\Gamma$, are always in $\mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$. It leads to our last assertion. □

We know that the set of m-derivations of a Lie algebra A forms a Lie algebra by the following definition of its bracket of two elements $[f, g] = f \circ g - g \circ f$, then two consequences of the previous theorem are the following:

Corollary 2.29. *The Lie algebra of $(m > 2)$-derivations of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ denoted by $Der^m(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) = Der(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) + \mathcal{L}(A_1, A_2)$ where $\mathcal{L}(A_1, A_2)$ notes the set of linear mappings from A_1 to A_2 and $Der^2 = Der$. If the connection is flat, $Der^m(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) = Der(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$ for all $m \geq 2$.*

Corollary 2.30. *The algebra $Der^m(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) = Der(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$ for all $m \geq 2$ if and only if the connection is flat.*

Proof. If the connection is flat, we have the result by the previous corollary. Conversely, if $Der^m(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) = Der(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$ for all $m \geq 2$, then $\mathcal{L}(A_1, A_2) \subset Der(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$. So $\mathcal{L}(A_1, A_2) \subset Der(A_1, A_2)$. It yields $\mathcal{L}(A_1, A_2) = \{0\}$ by the Proposition 2.22. It's not possible without nullity of A_1, then $k = 0$ everywhere. By Corollary 2.16, the connection is flat. □

Remark 2.31. The elements of $\mathcal{N}_0 \ominus A_2$ are locally of the form:

$$\underset{1 \leq i \leq n}{X^i(\overset{\wedge}{y^t}, k+1 \leq t \leq n)\frac{\partial}{\partial x^i}} + \underset{1 \leq j \leq k}{X'^j(\overset{\wedge}{y^t}, k+1 \leq t \leq n)\frac{\partial}{\partial y^j}}$$

where $X(\overset{\wedge}{y^t})$ means that expression of X is without y^t.

In the three following assertions, $\oplus$ means a direct sum of modules. The first space of cohomology of Chevalley-Eilenberg of a Lie algebra A, $H^1(A)$ is the quotient of vector spaces $Der(A)/adj_A$ where $Der(A)$ the Lie algebra of derivations of A and adj_A the Lie algebra of Lie derivatives with respect to elements of A. One result about $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ when $R = 0$ is in [11], that is to say $H^1(\mathfrak{A}_\Gamma) = 0$. Here, we find more general result by another way.

Corollary 2.32. *If $\mathfrak{A}_\Gamma^h$ is non-vanishing everywhere resp. $\mathfrak{A}_\Gamma^h = \{0\}$ and the rank of A_1 is a constant $k > 0$, then the first space of cohomology of Chevalley-Eilenberg of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is isomorphic to*

$$\left(\pi_*\left(\mathcal{N}_\Gamma\right)/\pi_*\left(\mathfrak{A}_\Gamma^h\right)\right) \otimes \left(\mathrm{End}\left(A_1\right)/adj_{\mathfrak{A}_\Gamma^v} \oplus \left(\left(\mathcal{N}_0 \ominus \mathfrak{A}_\Gamma^h\right)/A_2\right) \oplus \left(\mathrm{H}_R^1\left(\mathrm{B}\right) \otimes \mathbb{R}\right)^k\right)$$

resp. isomorphic to

$$\left(\mathrm{End}\left(A_1\right)/adj_{\mathfrak{A}_\Gamma^v} \oplus \left(\left(\mathcal{N}_0 \ominus \mathfrak{A}_\Gamma^h\right)/A_2\right) \oplus \left(\mathrm{H}_R^1\left(\mathrm{B}\right) \otimes \mathbb{R}\right)^k\right)$$

where $\mathrm{H}_R^1(\mathrm{B})$ is the first space of cohomology of de Rham on the closed 1-differential forms relative to the foliation, from A_2 towards $F_1(\mathcal{T}M)$. If the constant $k = 0$, $H^1(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$ equals to 0.

Proof. Using the beginning of the proof of Theorem 2.28, the first space of cohomology of Chevalley-Eilenberg of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is isomorphic to the direct product of $\mathfrak{A}_\Gamma^h$ and $\mathfrak{A}_\Gamma^v$. If $\mathfrak{A}_\Gamma^h$ is non-null everywhere, then we know that the first is isomorphic to $\pi_*(\mathcal{N}_\Gamma)/\pi_*\left(\mathfrak{A}_\Gamma^h\right)$ where $\mathcal{N}_\Gamma$ is the normalizer of $\mathfrak{A}_\Gamma^h$ in $\chi(\mathcal{T}M)$ [11]. For that of $\mathfrak{A}_\Gamma^v$, we deduce the result from Theorem 2.28 for $m = 2$. That is to say, one part of the cohomological result $\mathrm{End}(A_1)/adj_{\mathfrak{A}_\Gamma^v} \oplus \left(\left(\mathcal{N}_0 \ominus \mathfrak{A}_\Gamma^h\right)/A_2\right)$ is immediate because $[A_1, A_1] = \{0\}$, $[A_2, A_1] = A_1$ and $\mathcal{N}_0$ is the normalizer of A_2 such that A_2 has only derivations like Lie derivatives with respect to elements of $\mathcal{N}_0$. For the second part one $\left(\mathrm{H}_R^1(\mathrm{B}) \otimes \mathbb{R}\right)^k$, we can admit that local definition of α^i $(1 \leq i \leq k)$ is invariant by change of adapted coordinates to the foliation, and $D^{\alpha^i} \circ D^{\alpha^j} = 0$ for all i, j. Thus, by (2) of Proposition 2.25 adapted to our derivation, by the result in the end of the adapted proposition, by the closeness and the condition of exactitude of the forms α and the result of [8], we can find the rest of cohomological result. As for the case where $\mathfrak{A}_\Gamma^h = \{0\}$, we use again Theorem 2.28 in order to arrive to $\mathrm{H}^1\left(\mathfrak{A}_\Gamma^v\right)$ and we conclude by the previous result. If $k = 0$, the curvature $R = 0$ by Corollary 2.16. So we can apply Theorem 4.4 of [11] for the final result. □

From this assertion, we can deduce immediately the following result

Corollary 2.33. *The normalizer of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is the sum $\mathcal{N}_\Gamma + \mathcal{N}_0$.*

As our study about $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R = \mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$ in [11], that is to say we have studied subsets of $\mathfrak{A}_\Gamma$ included in the horizontal space or in the vertical space. This next proposition suggests the study of subsets of $\mathfrak{A}_\Gamma$ which are neither contained in $\mathfrak{A}_\Gamma^h$ nor in $\mathfrak{A}_\Gamma^v$ nor in $\mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$.

Proposition 2.34. *Each m-derivation of $\mathfrak{A}_\Gamma$ stabilizes $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ and $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is a characteristic ideal of $\mathfrak{A}_\Gamma$. So we can define the quotient Lie algebra $\mathfrak{A}_\Gamma/(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$. Moreover, elements of $\mathfrak{A}_\Gamma \ominus (\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$ are known by the inclusion $\mathfrak{A}_\Gamma \ominus (\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) \subset \mathcal{N}_\Gamma + \mathcal{N}_0$.*

Proof. We know by Corollary 2.17 that the derivative ideal of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ remains $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ itself. As $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ equals to direct product of ideals $\mathfrak{A}_\Gamma^h$ and $\mathfrak{A}_\Gamma^v$, then $[\mathfrak{A}_\Gamma, \mathfrak{A}_\Gamma \cap \mathfrak{N}_R] \subset \mathfrak{A}_\Gamma \cap \mathfrak{N}_R$. The result comes from the previous one by using relation (2.7) and Corollary 2.33. □

Comparing calculations of elements of $\mathfrak{A}_\Gamma^v$ by $\mathfrak{A}_{H^0}$ and by H^0 cf. Proposition 2.8, the first "$\mathfrak{A}_\Gamma^v$ is the centralizer of $\mathfrak{A}_{H^0}$" allows us to see the previous theoretical results, the second "the set of commutators of H^0 is $\mathfrak{A}_\Gamma^v$" is especially practical for calculation of these elements on Maple software cf. examples in the second section. We can state again on $\mathfrak{A}_{H^0}$:

Proposition 2.35. *Each m-derivation of $\mathfrak{A}_{H^0}$ can be split into $D = D^{22} + D^{21} + D^{12} + D^{11}$ where D^{22} the linear part of D from B_2 towards B_2, D^{21} from B_2 to B_1, D^{12} to B_1 towards B_2 and D^{11} of B_1 to B_1, where $\mathfrak{A}_{H^0} = B_1 + B_2$ by (2.5) in the order of the splitting, with $[B_1, B_1] = B_1$, $[B_1, B_2] = B_2$, $[B_2, B_2] = \{0\}$. Furthermore, these D^{ij} are m-derivations.*

Proof. We can reason similarly like in the proof of Proposition 2.23. □

Proposition 2.36. *The normalizer $\mathcal{N}_2$ of $\mathfrak{A}_{H^0}$ is locally the sum of the module generated in $F(U)$ by $\frac{\partial}{\partial x^t}$, $\frac{\partial}{\partial y^i}$, $y^l \frac{\partial}{\partial y^j}$ for all $1 \leq l, j, i \leq k$, $1 \leq t \leq n$ and the module $\underset{k+1 \leq j \leq n}{X^j(y^t, k+1 \leq t \leq n)\frac{\partial}{\partial y^j}}$. Then $\mathcal{N}_2$ contains strictly $\mathfrak{A}_{H^0}$.*

Proof. It's just a calculation from definition of normalizer. □

Theorem 2.37. *The derivative ideal of* $\mathfrak{A}_{H^0}$ *is* $\mathfrak{A}_{H^0}$. *The* $(m>2)$*-derivation of* $\mathfrak{A}_{H^0}$ *is a sum of endomorphism of* B_2*, of a linear mapping from* B_2 *to* B_1*, of a Lie derivative with respect to an element of* B_2*, of a derivation of the form* $D^{\alpha,0}$ *analogous to that of Proposition 2.26 and a Lie derivative with respect to an element of the normalizer* $\mathcal{N}_1$ *of* B_1 *in* $\chi(\mathcal{T}M)$. *Its first cohomology of Chevalley-Eilenberg's space is isomorphic to* $\left((\mathrm{End}\,(B_2))\,/adj_{\mathfrak{A}_{H^0}}\right)\oplus\left(\mathrm{H}^1_R\,(\mathrm{B}')\otimes\mathbb{R}\right)^k\oplus(\chi(\mathbb{R}^n))$ *if* $\mathfrak{A}_{H^0}$ *is of constant rank equals to* $n+k$ *(*B' *is the set of closed 1-differential forms relative to the foliation, from* B_1 *towards* $F(U)$*),* $\oplus$ *notes a direct sum of modules. Furthermore* $Der^m(\mathfrak{A}_{H^0})=Der(\mathfrak{A}_{H^0})+\mathcal{L}(B_2,B_1)$. *In the case where* $R=0$*, all* m*-derivations of* $\mathfrak{A}_{H^0}$ *are inner with respect to elements of its normalizer* $\mathcal{N}_2$ *which is isomorphic to* $\chi(M)\otimes\chi(\mathbb{R}^n)$. *In all cases,* $H^1(\mathfrak{A}_{H^0})\neq 0$.

Proof. The first assertion comes directly from Proposition 2.35. The next, it is sufficient to use Proposition 2.35 and to reason similarly as the part $\mathfrak{A}^v_\Gamma$ of Theorem 2.28 and of Corollary 2.32. The next result is deduced from the previous one where $\mathcal{N}_1$ is locally

$$\underset{1\le i\le n}{X^i\frac{\partial}{\partial x^i}}+\underset{1\le j\le n}{X'^j(y^t,1\le t\le n)\frac{\partial}{\partial y^j}}\text{ where the }X^i\in F(U). \tag{2.9}$$

The hypothesis of the next assertion suggests that $k=0$ everywhere. So that, $B_2=\{0\}$ and the corresponding $D^{\alpha,0}$ are null. It yields all m-derivations of $\mathfrak{A}_{H^0}$ to be those of B_1. By these results, we can conclude that $H^1(\mathfrak{A}_{H^0})\neq 0$. □

We reason in the same way as Corollary 2.30 to obtain,

Corollary 2.38. *We have* $Der^m(\mathfrak{A}_{H^0})=Der(\mathfrak{A}_{H^0})$ *for all* $m\ge 2$ *if and only if the connection is flat.*

3. Results on A_Γ, $\mathfrak{N}_R$ by local methods and examples

We take a coordinates system $(x^i)_{i=1,\dots,2n}$ on $\mathcal{T}M$ where $\Gamma=\Gamma^j_i dx^i\otimes\frac{\partial}{\partial x^j}$. We have followed the local formulas defining elements of A_Γ [7] in [11]. Coming back to this definition, we find that a vector field $X=X^i\frac{\partial}{\partial x^i}$ is in A_Γ if and only if it satisfies to a system of $4n^2$ linear equations of partial derivatives:

$$X^i\frac{\partial\Gamma^j_k}{\partial x^i}+\Gamma^j_i\frac{\partial X^i}{\partial x^k}-\Gamma^i_k\frac{\partial X^j}{\partial x^i}=0. \tag{3.1}$$

This rectification doesn't modify the results in [11] because our calculus base only on the vector form of the connection regardless of partial differential equations defining $\mathfrak{A}_\Gamma$.

Remark 3.1. Certains theorems in this section was made by the authour in 2016 in a preprint in the University of Antananarivo, Madagascar independently of the results of [4] and [5].

We try to know more the elements of $\mathfrak{A}_\Gamma$. So,

Proposition 3.2. *Each element* X *of* $\mathfrak{A}_\Gamma$ *is locally* $X=\underset{1\le i\le 2n}{X^i\frac{\partial}{\partial x^i}}$ *where* $\Gamma=dx^i\otimes\otimes\frac{\partial}{\partial x^i}-2\Gamma^i_j dx^j\otimes\frac{\partial}{\partial y^i}-dy^i\otimes\frac{\partial}{\partial y^i}$ *(*$x^{n+i}=y^i$ *and* $\Gamma^{n+i}_j=\Gamma^i_j$ *for* $1\le i\le n$*) verifies:*

$$\frac{\partial X^j}{\partial x^i}=0,\mathit{for}\ n+1\le i\le 2n\ \mathit{and}\ 1\le j\le n, \tag{3.2}$$

$$\frac{\partial X^j}{\partial x^k}+\underset{1\le i\le 2n}{X^i\frac{\partial\Gamma^j_k}{\partial x^i}}+\underset{1\le i\le n}{\Gamma^j_{i,i\neq j}\frac{\partial X^i}{\partial x^k}}-\underset{n+1\le i\le 2n}{\Gamma^{i,i\neq k}_k\frac{\partial X^j}{\partial x^i}}=0\ \mathit{for}\ n+1\le j\le 2n\ \mathit{and}\ 1\le k\le n. \tag{3.3}$$

Proof. We revisit the equations (3.1) taking into account the above local expression of Γ and distinguishing the cases $j=k$ and $j\neq k$. □

Another proof of Proposition 3.9 of [11] will be,

Corollary 3.3. *All vector fields of* $\mathfrak{A}_\Gamma$ *are projectable.*

Proof. We check the relation (3.2) which says directly that the vector field in question is projectable. □

By Proposition 3.2, we reduce the system (3.1) of $4n^2$ partial differential equations which elements of $\mathfrak{A}_\Gamma$ are defined. It is a system of n^2 partial differential equations of first order such that the above elements are projectable. We can again deduce from this proposition and know more the elements of $\mathfrak{A}_\Gamma^v$ by:

Corollary 3.4. *Each element X of $\mathfrak{A}_\Gamma^v$ whose its local form is $\underset{1\leq i\leq n}{X^i\frac{\partial}{\partial y^i}}$ verifies:*

$$\frac{\partial X^j}{\partial x^k}+\underset{1\leq i\leq n}{X^i\frac{\partial \Gamma_k^j}{\partial y^i}}-\underset{1\leq i\leq n}{\Gamma_k^i\frac{\partial X^j}{\partial y^i}}=0 \text{ for } 1\leq j\leq n \text{ and } 1\leq k\leq n. \tag{3.4}$$

Each element X of $\mathfrak{A}_\Gamma^h$ whose its local form is $\underset{1\leq i\leq n}{X^i\frac{\partial}{\partial x^i}}-\Gamma_i^t X^i\frac{\partial}{\partial y^t}$ satisfies:

$$-X^i\frac{\partial \Gamma_i^j}{\partial x^k}+X^i\frac{\partial \Gamma_k^j}{\partial x^i}-\Gamma_t^i X^t\frac{\partial \Gamma_k^j}{\partial y^i}+\Gamma_k^i X^t\frac{\partial \Gamma_t^j}{\partial y^i}=0 \text{ for } 1\leq j\leq n \text{ and } 1\leq k\leq n. \tag{3.5}$$

In the following, we have a chart $\mathcal{T}U$ with local coordinates system $(x^i,y^j)_{1\leq i,j\leq n}$ of $\mathcal{T}M$.

Proposition 3.5. *If $X\in\mathfrak{A}_\Gamma-\{0\}$ with $X\notin\mathfrak{A}_\Gamma\cap\mathfrak{N}_R$ and $X=X^i\frac{\partial}{\partial x^i}+X'^i\frac{\partial}{\partial y^i}$ in local coordinates of $\mathcal{T}U$, then it exists locally mappings between vertical components of a vector field of $\mathfrak{A}_\Gamma-(\mathfrak{A}_\Gamma\cap\mathfrak{N}_R)$ with another components. Moreover, if there is i_0 such that $\Gamma_{i_0}^j=0$ for all j, X^{i_0} depends at least on another component of X.*

Proof. Suppose $X\in\mathfrak{A}_\Gamma-\{0\}$ with $X\notin\mathfrak{A}_\Gamma\cap\mathfrak{N}_R=\mathfrak{A}_\Gamma^h\otimes\mathfrak{A}_\Gamma^v$. We reason in local coordinates $(x^i,y^j)_{1\leq i,j\leq n}$ of $\mathcal{T}U$. Let $X\in\mathfrak{A}_\Gamma-(\mathfrak{A}_\Gamma\cap\mathfrak{N}_R)$ with $X=X^i\frac{\partial}{\partial x^i}+X'^i\frac{\partial}{\partial y^i}$ on $\mathcal{T}U$. Suppose that there's j_0 such that for all $j\neq j_0$ and i, $X'^{j_0}\left(\overset{\wedge}{X^i},\overset{\wedge}{X'^j}\right)\neq 0$. Then on $\mathcal{T}U$, $X'^{j_0}\frac{\partial}{\partial y^{j_0}}\in\mathfrak{A}_\Gamma^v-\{0\}$ by (3.4), it is a contradiction. In the same way, if we suppose that it exists i_0 with all $i\neq i_0$ and j, $X^{i_0}\left(\overset{\wedge}{X^i},\overset{\wedge}{X'^j}\right)\neq 0$. Then $X^{i_0}\frac{\partial}{\partial x^{i_0}}\in\mathfrak{A}_\Gamma^h-\{0\}$ by (3.5), it is again a contradiction. □

Definition 3.6. The complete lift of a X in $\chi(M)$ on the bundle TM is noted by $\overline{X}$. Locally on TM, $(x^i,y^j)_{1\leq i,j\leq n}$ where $X=X^i\frac{\partial}{\partial x^i}\in\chi(M)$, we have

$$\overline{X}=X^i\frac{\partial}{\partial x^i}+y^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i} \text{ with } i,j=1,\dots,n.$$

The complete lift of $\chi(M)$ is denoted by $\overline{\chi(M)}$. It is easy to extend in the natural way the isomorphism f in the proof of Proposition 3.16 of [11] from $\chi(M)$ to $\overline{\chi(M)}$. We will note G this extension of isomorphism.

Definition 3.7. A strong torsion T of Γ is a 1-vector form

$$i_S t-\frac{1}{2}[C,\Gamma]$$

where S a semi-spray associated to Γ cf. [3]. A semi-spray is a vector field on $\mathcal{T}M$ such that $JS=C$ with C the canonical field on TM (C is equally the Liouville field), $t=\frac{1}{2}[J,\Gamma]$ is the weak torsion of Γ and i the inner product.

A semi-spray in local coordinates is

$$y^i\frac{\partial}{\partial x^i}-2G^j\frac{\partial}{\partial y^j},$$

where the G^j are C^∞ of $\mathcal{T}M$, $C=y^i\frac{\partial}{\partial y^i}$.
Let us recall a theorem of decomposition of connections in [3]:

Theorem 3.8. *Let S be a semi-spray and T be a semi-basic 1-vector form in balance with S cf.* [3]*, then it exists one and only one connection Γ such that S is its semi-spray and T its strong torsion with $\Gamma = [J, S] + T$.*

Now we recall a Lie sub-algebra of $\mathfrak{A}_\Gamma$, it is the intersection of $\mathfrak{A}_\Gamma$ with the complete lift $\overline{\chi(M)}$ of $\chi(M)$, denoted $\overline{\mathfrak{A}_\Gamma}$ and called the Lie algebra of infinitesimal automorphisms of Γ. This Lie algebra has been studied in [7] for a particular Grifone's connection $[J, S]$ of weak torsion free with S an arbitrary semi-spray. Here, we give our contributions on study of $\overline{\mathfrak{A}_\Gamma}$ for a general connection of Grifone, that is to say, its two torsions aren't necessarily null.

Theorem 3.9. *Locally $X = X^i \frac{\partial}{\partial x^i} + y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial y^j}$ is in $\overline{\mathfrak{A}_\Gamma}$ if and only if all the X^j depend only on the x^t and*

$$y^t \frac{\partial^2 X^j}{\partial x^k \partial x^t} + X^t \frac{\partial \Gamma^j_k}{\partial x^t} + y^u \frac{\partial X^t}{\partial x^u} \frac{\partial \Gamma^j_k}{\partial y^t} + \Gamma^j_t \frac{\partial X^t}{\partial x^k} - \Gamma^t_k \frac{\partial X^j}{\partial x^t} = 0, \tag{3.6}$$

in particular if Γ is homogeneous of degree 1 or linear connection of Grifone, that is to say the $\Gamma^j_k = y^u \gamma^j_{ku}$

$$\frac{\partial^2 X^j}{\partial x^k \partial x^l} + X^t \frac{\partial \gamma^j_{kl}}{\partial x^t} + \frac{\partial X^t}{\partial x^l} \gamma^j_{kt} + \gamma^j_{tl} \frac{\partial X^t}{\partial x^k} - \gamma^t_{kl} \frac{\partial X^j}{\partial x^t} = 0, \tag{3.7}$$

where t, u run over 1 to n; $j, k, l \in [1, n]$.

Proof. It is a consequence of Proposition 3.2 and of definition of $\overline{\mathfrak{A}_\Gamma}$. □

A generalization of corollary of the Proposition 11 of [7] can be stated as follow

Corollary 3.10. *If the connection is homogeneous of degree $l \in \mathbb{Z} - \{0\}$ ($[C, \Gamma] = l\Gamma$) then $\overline{\mathfrak{A}_\Gamma}$ is locally a Lie sub-algebra of affine vector fields of $\mathbb{R}^{2n}$, in this case $\dim(\overline{\mathfrak{A}_\Gamma}) \leq n^2 + n$.*

Proof. The homogeneity of a connection is characterized by the homogeneity plus one in the y^i of local coordinates of all local coefficients Γ^k_j of the connection Γ. The proof uses the following, it is sufficient to remark that locally $X \in \overline{\mathfrak{A}_\Gamma}$ with all its non-vertical components X^j depend only on x^i and verify

$$\frac{\partial^2 X^j}{\partial x^k \partial x^t} = 0$$

for all j, k, t by (3.6). □

Theorem 3.11. *If we suppose $G^{-1}\left(\overline{\mathfrak{A}_\Gamma}\right)$ contains locally all constant vector fields and the Euler vector field for a homogeneous connection of degree $l \in \mathbb{Z} - \{0\}$, then all m-derivations of $\overline{\mathfrak{A}_\Gamma}$ are derivations and are Lie derivatives with respect to elements of its normalizer $\mathcal{N}$, thus $H^1\left(\overline{\mathfrak{A}_\Gamma}\right) = \mathcal{N}/\overline{\mathfrak{A}_\Gamma}$. Each $(m \geq 2)$-derivation of $\mathcal{N}$ is inner and $H^1(\mathcal{N}) = 0$. Suppose that the maximum dimension is reached by $\overline{\mathfrak{A}_\Gamma}$ in whole TM then $H^1\left(\overline{\mathfrak{A}_\Gamma}\right) = 0$*

Proof. If we have such hypothesis for $\overline{\mathfrak{A}_\Gamma}$ then in local coordinates on $(x^i, y^j)_{1 \leq i,j \leq n}$, $\overline{\mathfrak{A}_\Gamma}$ contains locally the space generated by the $\frac{\partial}{\partial x^j}$ (constant vector fields) and $\underset{1 \leq i \leq n}{x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}}$ (Euler field on TM with $\underset{1 \leq i \leq n}{x^i \frac{\partial}{\partial x^i}}$ that on M) where the j runs from 1 to n, and all elements of $\overline{\mathfrak{A}_\Gamma}$ are locally affine cf. Corollary 3.10. By [10], each m-derivation of $G^{-1}\left(\overline{\mathfrak{A}_\Gamma}\right)$ is a derivation and a Lie derivative with respect to an element of $G^{-1}\left(\overline{\mathfrak{A}_\Gamma}\right)$. By isomorphism, each m-derivation of $\overline{\mathfrak{A}_\Gamma}$ is a derivation and is a Lie derivative by one element of normalizer $\mathcal{N}$ of $\overline{\mathfrak{A}_\Gamma}$ and $H^1\left(\overline{\mathfrak{A}_\Gamma}\right) = \mathcal{N}/\overline{\mathfrak{A}_\Gamma}$. Because $\mathcal{N}$ is itself locally affine and contains all vector fields locally of form $x^{i_0} \frac{\partial}{\partial x^{i_0}}$ with i_0 fixed between 1 and n, the Corollary 3.12 of [10] permits to say that all m-derivations of $\mathcal{N}$ are inner, thus $H^1(\mathcal{N}) = 0$. If the dimension $n^2 + n$ of $\overline{\mathfrak{A}_\Gamma}$ is reached for all points of TM, then $G^{-1}\left(\overline{\mathfrak{A}_\Gamma}\right)$ is locally the Lie algebra of affine vector fields of $\mathbb{R}^n$, its normalizer remains itself, then $H^1\left(\overline{\mathfrak{A}_\Gamma}\right) = 0$ by the previous results. □

In the following four examples, the considered smooth manifold for each connection Γ is $\mathbb{R}^3$ with $T\mathbb{R}^3$ of system of coordinates $(x^1, x^2, x^3, y^1, y^2, y^3)$.

Example 3.12. If we take the coefficients Γ^i_j null for all i, j, then we reach the maximum dimension 12 of $\overline{\mathfrak{A}_\Gamma}$ which is the Lie algebra spanned by all the $\frac{\partial}{\partial x^i}, x^j\frac{\partial}{\partial x^i} + +y^j\frac{\partial}{\partial y^i}$. By our theorem, each m-derivation of $\overline{\mathfrak{A}_\Gamma}$ is inner, in other words a Lie derivative with respect to a vector field of $\overline{\mathfrak{A}_\Gamma}$, thus $H^1\left(\overline{\mathfrak{A}_\Gamma}\right) = 0$.

Example 3.13. Taking the coefficients of Γ such that $\Gamma^1_2 = 1$ and null otherwise, we have $\overline{\mathfrak{A}_\Gamma}$ is of dimension 8: $\underset{1\le i\le 3}{\frac{\partial}{\partial x^i}}, \underset{1\le i\le 2}{x^i\frac{\partial}{\partial x^i}} + \underset{1\le j\le 2}{y^j\frac{\partial}{\partial y^j}}, x^2\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial y^1}, x^3\frac{\partial}{\partial x^1} + +y^3\frac{\partial}{\partial y^1}, x^2\frac{\partial}{\partial x^3} + y^2\frac{\partial}{\partial y^3}, x^3\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial y^3}$. By our theorem, each m-derivation of this Lie algebra is a Lie derivative by an element of its normalizer $\mathcal{N}$ which is $\overline{\mathfrak{A}_\Gamma}$ plus the vector space V spanned by $x^1\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial y^1}, x^2\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial y^2}$. Thus, $H^1\left(\overline{\mathfrak{A}_\Gamma}\right) = V$ and $H^1(\mathcal{N}) = 0$.

Remark 3.14. If the hypotheses of Theorem 3.11 are non-satisfied, then it can exist a non-inner derivation of $\overline{\mathfrak{A}_\Gamma}$ with respect of its normalizer as in the end of Example 3.15.

Example 3.15. A Grifone connection is such that:

$$\Gamma^1_1 = e^{x^3}, \Gamma^1_2 = e^{x^3}$$

and null otherwise. The non-null coefficients up to an antisymmetry of the curvature are

$$R^1_{13} = -e^{x^3}, R^1_{23} = -e^{x^3}.$$

Then $X = \underset{1\le i\le 3}{X^i\frac{\partial}{\partial x^i}} + \underset{1\le j\le 3}{X'^j\frac{\partial}{\partial y^j}} \in \mathfrak{N}_R$ is such that $X^1\frac{\partial}{\partial x^1} - X^1\frac{\partial}{\partial x^2} + \underset{1\le j\le 3}{X'^j\frac{\partial}{\partial y^j}}$ where $X^1, X'^j \in F(T\mathbb{R}^3)$. In that case, the nullity space of the curvature is involutive, it confirms Proposition 2.4. The horizontal space of the nullity of the curvature is the Lie algebra

$$\mathfrak{N}^h_R = \left\{X^1\frac{\partial}{\partial x^1} - X^1\frac{\partial}{\partial x^2}\right\}.$$

As for $X \in \mathfrak{A}^h_\Gamma$,

$$\mathfrak{A}^h_\Gamma = \left\{X^1(x^i, i = 1, 2, 3)\frac{\partial}{\partial x^1} - X^1(x^i, i = 1, 2, 3)\frac{\partial}{\partial x^2}\right\}.$$

The centralizer of $\mathfrak{A}_{H^0} = \left\langle \frac{\partial}{\partial x^1} - e^{x^3}\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2} - e^{x^3}\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^3}, e^{x^3}\frac{\partial}{\partial y^1}\right\rangle_{F(\mathbb{R}^3)}$ or the set of all commutators of H^0 is $\mathfrak{A}^v_\Gamma$ cf. Proposition 2.8 whose elements are easy to find by calculations by Maple from this proposition or by the relation (3.4), that is to say the $\underset{i=1,2,3,}{Y^i\frac{\partial}{\partial y^i}}$ such that

$$Y^i(y^2, y^3).$$

The Lie algebra $\overline{\mathfrak{A}_\Gamma}$ is a vector space generated by $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$ which is a Lie sub-algebra of affine vector fields of $\mathbb{R}^6$, of dimension 2 less than or equal to $3^2 + 3 = 12$, by Corollary 3.10. The Lie algebra $\overline{\mathfrak{A}_\Gamma}$ contains a derivation which is not a Lie derivative with respect to an element of its normalizer. For example, the linear mapping D defined by $D\left(\frac{\partial}{\partial x^2}\right) = \frac{\partial}{\partial x^1}$ and $D\left(\frac{\partial}{\partial x^1}\right) = 0$.

Example 3.16. The connection is such that:

$$\Gamma^1_1 = e^{x^3}y^1, \Gamma^1_2 = e^{x^3}$$

and null otherwise. The non-null coefficients of curvature up to an antisymmetry are

$$R^1_{12} = e^{2x^3}, R^1_{13} = -y^1e^{x^3}, R^1_{23} = -e^{x^3}.$$

Thus $X = \underset{1\le i\le 3}{X^i\frac{\partial}{\partial x^i}} + \underset{1\le j\le 3}{X'^j\frac{\partial}{\partial y^j}} \in \mathfrak{N}_R$ is of the form $X^1\frac{\partial}{\partial x^1} - y^1X^1\frac{\partial}{\partial x^2} - e^{x^3}X^1\frac{\partial}{\partial x^3} + +\underset{1\le j\le 3}{X'^j\frac{\partial}{\partial y^j}}$ with $X^1, X'^j \in F(T\mathbb{R}^3)$. In this case, each non-null horizontal vector field of $\mathfrak{N}_R$ is non-projectable, so $\mathfrak{A}^h_\Gamma = \{0\}$ and $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R = \mathfrak{A}^v_\Gamma$. The Lie algebra $\mathfrak{A}_{H^0}$ is

$$\left\langle \frac{\partial}{\partial x^1} - e^{x^3}y^1\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2} - e^{x^3}\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^3}, e^{x^3}y^1\frac{\partial}{\partial y^1}, e^{x^3}\frac{\partial}{\partial y^1}\right\rangle_{F(\mathbb{R}^3)}.$$

The space of nullity of R is not a Lie algebra because $\left[\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^1} - y^1\frac{\partial}{\partial x^2} - e^{x^3}\frac{\partial}{\partial x^3}\right] = -\frac{\partial}{\partial x^2} \notin \mathfrak{N}_R$, however the two vector fields in the bracket are in $\mathfrak{N}_R$. The horizontal nullity space of the curvature is the Lie algebra

$$\mathfrak{N}_R^h = \left\{X^1\frac{\partial}{\partial x^1} - y^1X^1\frac{\partial}{\partial x^2} - e^{x^3}X^1\frac{\partial}{\partial x^3}\right\}.$$

Here, we will have $\mathfrak{A}_\Gamma^v = \{\sum_{i=2,3} X'^i(y^2,y^3)\frac{\partial}{\partial y^i}\}$ and the Lie algebra $\mathfrak{A}_\Gamma \oplus \mathfrak{N}_R^h$ as a direct sum of modules. We observe that Proposition 3.14 of [11] doesn't work in the present example because $\mathfrak{A}_\Gamma^h = \{0\}$, however $\mathfrak{A}_\Gamma^v \neq \mathfrak{A}_\Gamma$ because $\frac{\partial}{\partial x^2} \in \mathfrak{A}_\Gamma - \mathfrak{A}_\Gamma^v$.

Example 3.17. The connection is such that $\Gamma_2^1 = \frac{1}{2}y^2e^{x^1}$ null otherwise. We find that $\mathfrak{A}_\Gamma^h$ is the $F\left(\mathbb{R}^3\right)$-module spanned by $\frac{\partial}{\partial x^3}$. For $X = \sum_{1\leq j\leq 3} X^j\frac{\partial}{\partial y^j} \in \mathfrak{A}_\Gamma^v$:

$$X^3\left(y^2,y^3\right) \text{ arbitrary, } X^1 = (y^2)^u y^1 F_1\left(y^3\right) + F_2\left(y^2,y^3\right) \text{ the } F_j \text{ arbitrary, } u\in\mathbb{N}$$
$$X^2 = (y^2)^{u+1}F_1\left(y^3\right),$$

by Corollary 3.4.
The elements of the quotient Lie algebra $\mathfrak{A}_\Gamma/\left(\mathfrak{A}_\Gamma\cap\mathfrak{N}_R\right) = \mathfrak{A}_\Gamma/\left(\mathfrak{A}_\Gamma^h\otimes\mathfrak{A}_\Gamma^v\right)$ mentioned in Proposition 2.34 are of the form $X = \sum_{1\leq j\leq 2} X^j\frac{\partial}{\partial x^j} + X'^1\frac{\partial}{\partial y^1}$ by Proposition 3.2:

$$\begin{aligned} X'^1 = {} & C_1y^1 + (y^2)^{2+t}g_1(x^1,y^2) + g_2(x^1,x^2)y^2 + y^2g_3(x^1) - \\ & -y^2\int (y^2)^t\left((2+t)\,g_1(x^1,y^2) + y^2\frac{\partial g_1}{\partial y^2}\right)dy^2 + \\ & +y^2\int (y^2)^t g_1(x^1,y^2)dy^2 + y^2\int (y^2)^w g_4(y^2)dy^2, \\ X^2 = {} & g_5(x^2) - 2\int e^{-x^1}\left((y^2)^{1+t}\frac{\partial g_1}{\partial x^1} + \frac{\partial g_2(x^1,x^2)}{\partial x^1} + \frac{dg_3(x^1)}{dx^1}\right)dx^1 + \\ & +2\int e^{-x^1}\left(\int (y^2)^t\left((2+t)\frac{\partial g_1}{\partial x^1} + y^2\frac{\partial^2 g_1}{\partial y^2\partial x^1}\right)dy^2\right)dx^1 - \\ & -2\int e^{-x^1}\left(\int (y^2)^t\frac{\partial g_1}{\partial x^1}dy^2\right)dx^1, \\ X^1 = {} & \int 2e^{x^1}\frac{\partial^2 g_2}{\partial x^1\partial x^2}dx^1 - \frac{dg_5(x^2)}{dx^2} + C_1 - 2e^{-x^1}\frac{\partial g_2(x^1,x^2)}{\partial x^2}, \end{aligned}$$

where $C_1\in\mathbb{R}$, the g_i are arbitrary functions, $t,w\in\mathbb{N}$.

Remark 3.18. For a homogeneous of degree 1, $\overline{\mathfrak{A}_\Gamma}$ is not in general locally a Lie sub-algebra of the Lie algebra of affine vector fields in $\mathbb{R}^{2n}$ as the end of Example 3.19 shows.

Example 3.19. We can revisit Example 4.8 of [11], where we take $\mathbb{R}^4$ of coordinates system $(x^1,x^2,x^3,x^4,y^1,y^2,y^3,y^4)$ with Γ is such that $\Gamma_2^1 = -\frac{y^2}{2}e^{x^1}, \Gamma_1^2 = \frac{y^2}{2}, \Gamma_2^2 = \frac{y^1}{2}$. We recall that $R_{2,1}^1 = \frac{y^2}{2}e^{x^1}$, $R_{2,1}^2 = -\frac{y^1}{4}$, null otherwise up to an antisymmetry and

$$\mathfrak{N}_R = \left\{X^3\frac{\partial}{\partial x^3} + X^4\frac{\partial}{\partial x^4} + Y^i\frac{\partial}{\partial y^i}\right\},\ \mathfrak{N}_R^h = \left\{X^3\frac{\partial}{\partial x^3} + X^4\frac{\partial}{\partial x^4}\right\}.$$

Numerically in Maple by the help of Corollary 3.4,

$$\mathfrak{A}_\Gamma^h = \left\{X^3(x^t)\frac{\partial}{\partial x^3} + X^4(x^t)\frac{\partial}{\partial x^4}\right\}.$$

Let us calculate the Lie algebra $\mathfrak{A}_\Gamma^v$ by Corollary 3.4 on Maple. The elements $X = \sum_{1\leq j\leq 4} X'^j\frac{\partial}{\partial y^j} \in \mathfrak{A}_\Gamma^v$ are such that:

$$X'^j\left((y^1)^2 + (y^2)^2e^{x^1}, y^3, y^4\right) \text{ are arbitrary where } j \text{ from 3 to 4},$$

$$X'^i = y^i F_1\left((y^1)^2 + (y^2)^2 e^{x^1}, y^3, y^4\right) \text{ where } F_1 \text{ an arbitrary function, } i \text{ from 1 to 2.}$$

One can also find the previous result on Maple by the fact that it is the centralizer of $\mathfrak{A}_{H^0}$

$$\left\langle \frac{\partial}{\partial x^1} - \frac{y^2}{2}\frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^2} + \frac{y^2}{2}e^{x^1}\frac{\partial}{\partial y^1} - \frac{y^1}{2}\frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}, \frac{y^2}{4}e^{x^1}\frac{\partial}{\partial y^1} - \frac{y^1}{4}\frac{\partial}{\partial y^2} \right\rangle_{F(\mathbb{R}^4)}.$$

We use (3.7) Theorem 3.9 in order to have the Lie algebra of infinitesimal automorphisms of Γ to be the space generated by:

$$\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}, x^3\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial y^3}, x^4\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial y^4}, -2\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial y^2},$$

$$x^2\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial y^1} + \left(e^{-x^1} - \frac{1}{4}(x^2)^2\right)\frac{\partial}{\partial x^2} - \frac{1}{2}e^{-x^1}\left(y^2x^2e^{x^1} + 2y^1\right)\frac{\partial}{\partial y^2},$$

$$x^4\frac{\partial}{\partial x^3} + y^4\frac{\partial}{\partial y^3}.$$

4. Some properties of the set of connections and associated Lie algebras

Here, we look at properties of the set of connections of Grifone on M. The situations are the following, neither a sum nor a composition of two usual linear connections (the latter does not even exist) is a linear connection. Unlike for the Grifone connections, here are some remarkable properties that the usual linear connections cannot have in any case, these are the following properties:

Proposition 4.1. *The arithmetic mean of a non-empty family of Grifone connections is a Grifone connection, the composition of an odd number of Grifone connections is one. That is to say if $\Gamma_1, \dots, \Gamma_u$ (resp. $\Gamma_1, \dots, \Gamma_{2u+1}$) are such connections, then $\frac{\Gamma_1+\dots+\Gamma_u}{u}$ (resp. $\Gamma_1 \circ \dots \circ \Gamma_{2u+1}$) is a connection of Grifone.*

Proof. It is easy to check all the relations from the Definition 2.2 for these connections in question. □

This proposition generates us many connections from those already given. Note that Γ is always invertible because $\Gamma^2 = I$ where I is the identity mapping, the sum and composition of two connections are not a connection. The multiplication by a real number of a connection is a connection if and only if this number is equal to 1. We will note K the set of all connections of Grifone on M. Now, we are looking for some algebraic structures on K. We can build the smallest non-Abelian multiplicative group G containing K by the composition of mappings. Then, the group $(G, \circ)$ is composed by all connections, the identity which is $\Gamma \circ \Gamma$, $\Gamma_1 \circ \Gamma_2$ which is locally of the form

$$dx^i \otimes \frac{\partial}{\partial x^i} - 2\left(\Gamma_{1i}^j - \Gamma_{2i}^j\right) dx^i \otimes \frac{\partial}{\partial y^j} + dy^i \otimes \frac{\partial}{\partial y^i}, \tag{4.1}$$

where Γ_{1i}^j and Γ_{2i}^j are the respective coefficients corresponding to Γ_1 and Γ_2. This kind of elements will be noted by ∇. According to Proposition 4.1, the composition on the left or on the right of a connection with the latter is a connection.
Next, we propose to make the smallest Abelian $\mathfrak{G}$ group containing K by the sum of two applications. To do this, it must be for all $t \in \mathbb{Z}$, $t.\Gamma \in \mathfrak{G}$. Therefore, the following elements that are written locally:

$$-2\Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j},$$

are in $\mathfrak{G}$. This kind of elements will in turn be denoted by γ, besides the tangent structure $J \in \mathfrak{G}$. Moreover, there is a natural bijective correspondence between K and the set of all of these γ. We deduce that $(G \cup \mathfrak{G}, +, \circ)$ is a ring. Moreover, we can construct $(G \cup \mathfrak{G}, +, .)$ as a vector space on $\mathbb{R}$ by replacing the set $\mathbb{Z}$ above by $\mathbb{R}$. Thus, we establish the law $[,]$ defined by $[f, g] = f \circ g - g \circ f$ for all $f, g \in G \cup \mathfrak{G}$, $(G \cup \mathfrak{G}, +, ., [,])$ is the smallest $\mathbb{R}$-Lie algebra containing K for these corresponding laws. This is justified by the following multiplication table:

$[,]$	Γ_1	∇_1	γ_1
Γ_2	γ_3	γ_4	γ_5
∇_2	γ_6	0	0
γ_2	γ_7	0	0

We deduce that,

Proposition 4.2. *The vector subspace of all γ of $(G \cup \mathfrak{G}, +, .)$ is an ideal of Lie algebra $(G \cup \mathfrak{G}, +, ., [,])$. The derivative ideal $\mathfrak{D}$ of this last Lie algebra coincides with this vector subspace which is in turn a characteristic ideal of $G \cup \mathfrak{G}$. Then, all m-derivations of $G \cup \mathfrak{G}$ restricted to this derivative ideal take its values in this ideal. All endomorphisms of $\mathfrak{D}$ are m-derivations of $\mathfrak{D}$ and $G \cup \mathfrak{G}$ is solvable of order 2 (that is to say the derivative ideal of the derivative ideal is reduced to zero). In addition, the centralizer of $G \cup \mathfrak{G}$ is null.*

We can say immediately that

Proposition 4.3. *Let's be a family of u Grifone connections where u is a nonzero natural integer and for $t \in \mathbb{N}$ such that $2t + 1 \leq u$, we have*

$$h_{\frac{\Gamma_1 + \cdots + \Gamma_u}{u}} = \frac{h_1 + \cdots + h_u}{u}, \; v_{\frac{\Gamma_1 + \cdots + \Gamma_u}{u}} = \frac{v_1 + \cdots + v_u}{u}$$

where h or v indexed by i or by the arithmetic mean of the big gammas is the horizontal resp. vertical projector of Γ_i resp. of the arithmetic mean of the big gammas. More

$$\begin{aligned} h_{\Gamma_1 \circ \cdots \circ \Gamma_{2t+1}} &= 2^{2t} \left(h_1 \circ \cdots \circ h_{2t+1}\right) - 2^{-1} \prod_{w=1}^{2t+1} \Gamma_w - \\ &\quad - 2^{-1} \prod_{1 \leq w_1 < w_2 \leq 2t+1} \Gamma_{w_1} \circ \Gamma_{w_2} \ldots \\ &\quad - 2^{-1} \prod_{1 \leq w_1 < w_2 < \cdots < w_{2t} \leq 2t+1} \Gamma_{w_1} \circ \cdots \circ \Gamma_{w_{2t}}. \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} v_{\Gamma_1 \circ \cdots \circ \Gamma_{2t+1}} &= 2^{2t} \left(v_1 \circ \cdots \circ v_{2t+1}\right) + \frac{(-1)^2}{2} \prod_{w=1}^{2t+1} \Gamma_w + \\ &\quad + \frac{(-1)^3}{2} \prod_{1 \leq w_1 < w_2 \leq 2t+1} \Gamma_{w_1} \circ \Gamma_{w_2} \ldots \\ &\quad + \frac{(-1)^{2t+1}}{2} \prod_{1 \leq w_1 < w_2 < \cdots < w_{2t} \leq 2t+1} \Gamma_{w_1} \circ \cdots \circ \Gamma_{w_{2t}}. \end{aligned} \tag{4.3}$$

We now propose to see some Lie algebras defined by the types of operations indicated above for connections of Grifone, we denote by R_Γ the curvature of Γ:

Proposition 4.4. *We give a family of u connections with a natural integer $u \neq 0$, then $X \in \mathfrak{A}_{\frac{\Gamma_1 + \cdots + \Gamma_u}{u}}$ if and only if $\sum_{i=1}^{u} [X, \Gamma_i(Y)] = \sum_{i=1}^{u} \Gamma_i [X, Y]$ for all $Y \in \chi(\mathcal{T}M)$. In particular, $\bigcap_{i=1}^{u} \mathfrak{A}_{\Gamma_i} \subset \mathfrak{A}_{\frac{\Gamma_1 + \cdots + \Gamma_u}{u}}$. The curvature R of $\frac{\Gamma_1 + \cdots + \Gamma_u}{u}$ is*

$$R_{\frac{\Gamma_1 + \cdots + \Gamma_u}{u}} = \frac{1}{u^2} \left(\sum_{i=1}^{u} R_{\Gamma_i} - \frac{1}{4} \sum_{i,j=1, i<j}^{u} [\Gamma_i, \Gamma_j] \right).$$

Moreover, let $t \in \mathbb{N}$ with $2t + 1 \leq u$, $\bigcap_{i=1}^{2t+1} \mathfrak{A}_{\Gamma_i} \subset \mathfrak{A}_{\Gamma_1 \circ \cdots \circ \Gamma_{2t+1}}$ more precisely if locally $X = \underset{1 \leq i \leq 2n}{X^i \frac{\partial}{\partial x^i}} \in \mathfrak{A}_{\Gamma_1 \circ \cdots \circ \Gamma_{2t+1}}$ where the $\Gamma_w = dx^i \otimes \frac{\partial}{\partial x^i} - 2\Gamma^i_{wj} dx^j \otimes \frac{\partial}{\partial y^i} - -dy^i \otimes \frac{\partial}{\partial y^i}$ $(x^{n+i} = y^i$ and $\Gamma^{n+i}_{wj} = \Gamma^i_{wj}$ for $1 \leq i \leq n)$, then X verifies:

$$\frac{\partial X^j}{\partial x^i} = 0, \; \text{for } n+1 \leq i \leq 2n \text{ and } 1 \leq j \leq n, \tag{4.4}$$

$$\frac{\partial X^j}{\partial x^k}+\sum_{w=1}^{2t+1}(-1)^{w+1}\underset{1\leq i\leq 2n}{X^i\frac{\partial \Gamma^j_{wk}}{\partial x^i}}+\sum_{w=1}^{2t+1}(-1)^{w+1}\underset{1\leq i\leq n}{\Gamma^j_{wi,i\neq j}\frac{\partial X^i}{\partial x^k}}-\sum_{w=1}^{2t+1}(-1)^{w+1}\underset{n+1\leq i\leq 2n}{\Gamma^{i,i\neq k}_{wk}\frac{\partial X^j}{\partial x^i}}=0 \tag{4.5}$$

for $n+1\leq j\leq 2n$, $1\leq k\leq n$ *and*

$$R_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}=\left(\sum_{i\ odd,\ i=1}^{2t+1}R_{\Gamma_i}\right)-\left(\sum_{i\ even,\ i=2}^{2t}R_{\Gamma_i}\right)+R'_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}},$$

where R' *an antisymmetric 2-vector form locally in* $(x^q,y^v)_{1\leq q,v\leq n}$ *of the form*

$$R'_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}=\frac{1}{2}R'^i_{jk}dx^j\wedge dx^k\otimes\frac{\partial}{\partial y^i}$$

with

$$\begin{aligned} R'^i_{jk} &= \sum_{w<v}^{2t+1}(-1)^{w+v}\left(\Gamma^l_{wk}\frac{\partial\Gamma^i_{vj}}{\partial y^l}+\Gamma^l_{vk}\frac{\partial\Gamma^i_{wj}}{\partial y^l}\right)- \\ &\quad -\sum_{w<v}^{2t+1}(-1)^{w+v}\left(\Gamma^l_{wj}\frac{\partial\Gamma^i_{vk}}{\partial y^l}+\Gamma^l_{vj}\frac{\partial\Gamma^i_{wk}}{\partial y^l}\right), \end{aligned}$$

these Γ^l_{wj} *are local coefficients of the connection* Γ_w *where* $1\leq w\leq 2t+1$.

Proof. For the first assertion, just notice that X belongs to $\mathfrak{A}_{\frac{\Gamma_1+\cdots+\Gamma_u}{u}}$ if and only if $L_X\left(\frac{\Gamma_1+\cdots+\Gamma_u}{u}\right)=0$. Since the Lie derivative is $\mathbb{R}$-linear, we find the result and deduce the following assertion. We can check that $R=-\frac{1}{8}[\Gamma,\Gamma]$ for a connection Γ by respecting the formulas on the brackets of the following 1-vector forms: for all endomorphisms L, F of $\chi(\mathcal{T}M)$ and $X,Y\in\chi(\mathcal{T}M)$, we have

$$\begin{aligned} [L,F](X,Y) &= [LX,FY]+[FX,LY]+LF[X,Y]+FL[X,Y]-L[FX,Y]- \\ &\quad -F[LX,Y]-L[X,FY]-F[X,LY], \end{aligned} \tag{4.6}$$

$$[X,L](Y)=[X,LY]-L[X,Y] \text{ cf. [2].} \tag{4.7}$$

The following assertion follows from it.
We use the formulas (3.2), (3.3) and then (4.1) for all local coordinates to prove the statements: "for all $t\in\mathbb{N}$, we have $\bigcap_{i=1}^{2t+1}\mathfrak{A}_{\Gamma_i}\subset\mathfrak{A}_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}$" and the following. As for the last statement, we rely on the formulas (2.2) and (4.1). □

Remark 4.5. We find that we did not necessarily $\bigcap_{i=1}^{u}\mathfrak{N}_{R_{\Gamma_i}}\subset\mathfrak{N}_{R_{\frac{\Gamma_1+\cdots+\Gamma_u}{u}}}$ and $\bigcap_{i=1}^{2t+1}\mathfrak{N}_{R_{\Gamma_i}}\subset\mathfrak{N}_{R_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}}$.

However, by the Proposition 4.4 and the formulas of (4.6), we have a remarkable result for the Lie algebra $\mathfrak{A}_\Gamma\cap\mathfrak{N}_R$ of a family of Grifone connections:

Proposition 4.6. *Let a family* $\{\Gamma_i\}_{i=1,\ldots,u}$ *of Grifone connections, then*

$$\bigcap_{i=1}^{u}\left(\mathfrak{N}_{R_{\Gamma_i}}\cap\mathfrak{A}_{\Gamma_i}\right)\subset\left(\mathfrak{N}_{R_{\frac{\Gamma_1+\cdots+\Gamma_u}{u}}}\cap\mathfrak{A}_{\frac{\Gamma_1+\cdots+\Gamma_u}{u}}\right).$$

If all these Γ_i *where* $1\leq i\leq 2t+1$ *(*$t>0$*) are homogeneous of degree -1, then*

$$R_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}=\left(\sum_{i\ odd,\ i=1}^{2t+1}R_{\Gamma_i}\right)-\left(\sum_{i\ even,\ i=2}^{2t}R_{\Gamma_i}\right),$$

$\bigcap_{i=1}^{2t+1}\mathfrak{N}_{R_{\Gamma_i}}\subset\mathfrak{N}_{R_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}}$ *and*

$$\bigcap_{i=1}^{2t+1}\left(\mathfrak{N}_{R_{\Gamma_i}}\cap\mathfrak{A}_{\Gamma_i}\right)\subset\left(\mathfrak{N}_{R_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}}\cap\mathfrak{A}_{\Gamma_1\circ\cdots\circ\Gamma_{2t+1}}\right).$$

Remark 4.7. We can replace an arithmetic mean by a weighted average in the last six propositions without Proposition 4.2, so we find similar results with the introduction of the respective weights of the connections in the calculation.

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References

[1] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1960.

[2] A. Frölicher and A. Nijenhuis, Theory of vector-valued differential forms, Proc. Kon. Ned. Akad. A, **59**, 338-359, (1956).

[3] J. Grifone, *Structure presque-tangente et connexions I*, Ann. Inst. Fourier Grenoble, **22** (1972), n°1, pp.287-334.

[4] Herinantenaina R. S., Ravelonirina H. S. G., *On Lie Algebras defined by a vector-valued 1-form*, Palestine Journal of Mathematics, V.11, special issue N. 3, 298-307, 2022.

[5] Herinantenaina R. S., Ravelonirina H. S. G., *An almost-product structure on the foliated manifold*, Advances in Mathematics: Scientific Journal, 9, n.12, 11019-11041, 2020.

[6] R. Hermann, *On the accessibility problem in control theory*, Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, pp. 325-332 (1963), New York.

[7] J. Klein, *On Lie Algebras of vector fields defined by vector forms*, Colloquia Math, Societatis Janos Bolyai, 1987.

[8] P. Lecomte, On the infinitesimal automorphism of the vector bundle. *J. Math. Pures Appl.* **60** (1981), 229-239.

[9] N. Nakanishi, *The first cohomology groups of infinite dimensional Lie algebras*, Pub. RIMS. Kyoto Univ., **18** (1982), pp.1-16.

[10] P. Randriambololondrantomalala, *Les m-dérivations des algèbres de Lie de champs de vecteurs polynomiaux*, Afrika Matematika, v. 27, 2016.

[11] P. Randriambololondrantomalala, H.S.G. Ravelonirina and M. Anona, *Sur les algèbres de Lie associées à une connexion*, Canadian Mathematical Bulletin, **58** number 4, pp. 692-703, 2015.

[12] P. Randriambololondrantomalala, H.S.G. Ravelonirina and M. Anona, *Sur les algèbres de Lie d'une distribution et d'un feuilletage généralisé*, African Diaspora Journal of Mathematics, **10** number 2, pp. 135-144, 2010.

[13] P. Randriambololondrantomalala, *The m-derivations of distribution Lie algebras*, Journal of Generalized Lie Theory and Applications, 9:1, 2015.

[14] H.S.G. Ravelonirina, P. Randriambololondrantomalala and M. Anona, *Sur les algèbres de Lie d'un système de champs de vecteurs permutables*, Italian Journal of Pure and Applied Mathematics, N.**29**, pp. 163-174, 2012.

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